Direct Approximation for Modified Szasz Operators in Lp-Spaces

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Abstract

The study of mixed type operators with the combination of two basis functions which may be either same or different, is very vast. Just now, the exploration of the properties of mixed type operators are limited not only to the functions of bounded variation but also to the unbounded variation too in some spaces. Several Mathematicians have used the convergence rate, Moduli of smoothness etc. to get some direct, inverse and saturation results for various types of mixed operators in their study and seen very interesting conclusions. In this paper, we discuss about direct results for these operators having the same basis functions as Szasz basis function in the Lp-spaces.

Keywords: Direct Theorem; Hölder inequality; $K-$functional; Moduli of smoothness; Szasz-Durrmeyer operators.

1 Introduction

There are several mathematicians [1], [7], [8] who explained approximation in several manners. In this paper, we study for mixed type operators in the L_p spaces, in which for a function $f(y) \in L_p[0,\infty)$, there is defined

$$\|f(y)\|_p = \begin{cases} \{\int_0^\infty |f(y)|^p dy\}^{1/p}, & 1 \le p < \infty; \\ \sup_{y\in[0,\infty)} |f(y)|, & p = \infty. \end{cases}$$

Gupta et al. [6], [5] proposed several types of combined operators. The results obtained by Ditzian-Totic [2], [3] have played very important role in our study. In this series we contribute our mentioned summation-integral type operators named as modified Szasz operators χ_η for $f \in L_p[0,\infty)$ as

$$\chi_\eta(f;x) = \eta \sum_{j=0}^{\infty} \sigma_{\eta,j}(x) \int_0^\infty \sigma_{\eta,j}(y) f(y) dy, \tag{1.1}$$

where $\sigma_{\eta,j}(x)$ is Szasz basis function defined by

$$\sigma_{\eta,j}(x) = \frac{(\eta x)^j}{(j)!} e^{-\eta x}.$$

Since $\chi_\eta(1,x) = 1$, it is clear to show that $\chi_\eta(f,x)$ are linear positive operators and the rate of convergence for these operators is of order $O(\frac{1}{\eta})$. In this paper, we give exciting approximation results for our discussed linear positive operators (1.1). Many authors [4], [9] etc. have discussed earlier the properties for several operators and found global results. There are obtained some auxiliary results related to the mentioned operators and also established direct results as approximation estimates in terms of modulus of continuity.

Remark 1. *In this paper 'C' represents arbitrary constant having no same value everywhere.*

2 Moment estimation

In this section, we give some lemmas related to the moments and other estimates for the mentioned operators (1.1).

Lemma 1. *Let $f \in L_p[0,\infty)$; $p \in [1,\infty]$, then the following inequality holds*

$$\|\chi_\eta(f;.)\|_p \leq \|f\|_p \tag{2.1}$$

Lemma 2. *For $\delta_\eta^2(x) = \varphi^2(x) + \frac{1}{\eta}, \varphi(x) = \sqrt{x}$, our operators (1.1) give*

$$\begin{aligned} \chi_\eta(1;x) &= 1, \qquad \chi_\eta(y-x,x) = \frac{1}{\eta}, \\ \chi_\eta((y-x)^2;x) &\leq \frac{C\delta_\eta^2(x)}{\eta}, \qquad \chi_\eta(y-x)^4;x) \leq \frac{C\delta_\eta^4(x)}{\eta^2}. \end{aligned}$$

Lemma 3. *For the operators χ_η, we can find that $\chi_\eta(y^2;x) \leq Cx^2$.*

Lemma 4. *For $\varphi(x) = \sqrt{x}$ and $y < w < x$,*

$$\frac{|y-w|}{\varphi^2(w)} \leq |y-x| \left(\frac{1}{\varphi^2(x)} + \frac{1}{\varphi^2(y)} \right).$$

Lemma 5. *For $y < w < x$ and $x \in [0, \frac{1}{\eta})$, we can find*

$$\frac{|y-w|}{\varphi^2(w) + 1/\eta} \leq \frac{|y-x|}{\varphi^2(x) + 1/\eta}$$

Proof. For $y < w < x$, it is obvious that $|y-w| \leq |y-x|$ and $\varphi^2(w) + 1/\eta \geq \varphi^2(x) + 1/\eta$, which follows the required lemma. Also since $y < w < x$ and $x < 1/\eta$ then $h(x) = \frac{x}{\varphi^2(x)+1/\eta}$ increases in $[0, \eta^{-1})$. □

Lemma 6. *If $f' \in A.C._{loc}$ and $p \in [1, \infty)$, we can have $\|\varphi^2 \chi''_\eta f\|_p \leq C \|\varphi^2 f''\|_p$.*

Lemma 7. *If*

$$U_\eta(h; x) = \eta \sum_{j=1}^{\infty} \sigma_{\eta,j-1}(x) \int_0^\infty \sigma_{\eta,j}(y) h(y) dt$$

then $U_\eta((y-x)^2; x) = \frac{6}{\eta^2} + \frac{2}{\eta} x + (\eta - 1) x^2$.

Proof. From here

$$U_\eta(1; x) = \eta, U_\eta(y; x) = x + \frac{2}{\eta}, U_\eta(y^2; x) = x^2 + \frac{6}{\eta} x + \frac{6}{\eta^2}.$$

Therefore

$$\begin{aligned} U_\eta((y-x)^2; x) &= U_\eta(y^2; x) - 2x U_\eta(y; x) + x^2 U_\eta(1; x) \\ &= x^2 + \frac{6}{\eta} x + \frac{6}{\eta^2} - 2x(x + \frac{2}{\eta}) + x^2 \eta \\ &= \frac{6}{\eta^2} + \frac{2}{\eta} x + (\eta - 1) x^2. \end{aligned}$$

□

3 Main Result

In this section we give first an important theorem needed to obtain the direct results and after that our required direct theorem.

Theorem 1. *If* $\overline{D} = \{g \in L_p[0,\infty) : g' \in A.C._{loc}; g'', \varphi^2 g'' \in L_p[0,\infty)\}$ *then for* $g \in \overline{D}, \delta_\eta^2(x) = \varphi^2(x) + \frac{1}{\eta}, p \in [1,\infty)$, *then*

$$\|\chi_\eta\left(\int_x^y (y-u)g''du;x\right)\|_p \leq \frac{C}{\eta}\|\delta_\eta^2 g''\|_p.$$

Proof. We find the result of this theorem based on the proof separately at the extreme points for $p = 1$ and $p = \infty$. First we discuss for $p = \infty$. Therefore taking $x \in E_n = [\frac{1}{\eta}, \infty)$ where $\eta \in N$; however we have $\delta_\eta^2 \sim \varphi^2(x)$. Applying Lemmas 2, Lemma 4 and Hölder inequality, we obtain

$$\left|\chi_\eta\left(\int_x^y (y-u)g''du;x\right)\right| \leq \frac{C}{\eta}\|\varphi^2 g''\|_\infty.$$

Again, taking $x \in E_\eta^c = [0, \frac{1}{\eta})$, Lemma 2 and Lemma 5 provide

$$\begin{aligned}
&\left|\chi_\eta\left(\int_x^y (y-u)g''du;x\right)\right| \\
\leq\ & \frac{C}{\eta}\|\varphi^2 g''\|_\infty \left|\chi_\eta\left(\int_x^y \frac{|y-u|}{\varphi^2(u)+1/\eta}du;x\right)\right| \\
\leq\ & \frac{C}{\eta}\|\varphi^2 g''\|_\infty \left|\chi_\eta\left(\frac{(y-x)^2}{\varphi^2(x)+1/\eta};x\right)\right| \\
\leq\ & \frac{C}{\eta}\|\varphi^2 g''\|_\infty \frac{\varphi^2(x)}{\eta-1}.\frac{1}{\varphi^2(x)+1/\eta} \leq \frac{C}{\eta}\|\varphi^2 g''\|_\infty.
\end{aligned}$$

Therefore this lemma is true for $p = \infty$. Now we prove the rest result i.e. for $p = 1$. From the definition of norm

$$\begin{aligned}
&\left\|\chi_\eta\left(\int_x^y (y-u)g''(u)du;x\right)\right\|_1 \\
=\ & \int_0^\infty \left|\chi_\eta\left(\int_x^y (y-u)g''(u)du;x\right)\right| dx \\
=\ & \int_0^\infty \left|\eta\sum_{j=0}^\infty \sigma_{\eta,j}(x)\left(\int_0^x + \int_x^\infty\right)\sigma_{\eta,j}(y)(y-u)g''(u)dudy\right| dx \\
\leq\ & \eta\int_0^\infty |g''(u)|\left(\int_u^\infty\int_0^u - \int_0^u\int_u^\infty\right)(u-y)\sum_{j=0}^\infty \sigma_{\eta,j}(x)\times \\
& \sigma_{\eta,j}(y)dydxdu
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\eta}{2}\int_0^\infty |g''(u)| \times \\
&\quad \left|\sum_{j=0}^{\infty}\left(\int_u^\infty\int_0^u - \int_0^u\int_u^\infty\right)\sigma_{\eta,j}(y)d(u-y)^2\sigma_{\eta,j}(x)dx\right| du \\
&:= \frac{1}{2}\int_0^\infty |g''(u)|(J_1+J_2)du.
\end{aligned}
$$

To estimate J_1 and J_2, while integrating by parts we shall use the fact

$$\sigma'_{\eta,j}(y) = \eta[\sigma_{\eta,j-1}(y) - \sigma_{\eta,j}(y)],\, j \geq 1$$

and obtain

$$
\begin{aligned}
J_1 &\leq \eta u^2 + \eta\sum_{j=1}^{\infty}\int_0^u (y-u)^2\sigma_{\eta,j-1}(u)\sigma_{\eta,j}(y)dy \\
J_2 &\leq \eta\sum_{j=1}^{\infty}\int_u^\infty (y-u)^2\sigma_{\eta,j-1}(u)\sigma_{\eta,j}(y)dy.
\end{aligned}
$$

Using Lemma 7, we get

$$
\begin{aligned}
\frac{1}{2}(J_1+J_2) &\leq \eta u^2 + \eta\sum_{j=1}^{\infty}\int_0^\infty (y-u)^2\sigma_{\eta,j}(y)\sigma_{\eta,j-1}(u)dt \\
&\leq u^2 + \varphi^2(u) \leq C\delta_\eta^2(u).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left\|\chi_\eta\left(\int_x^y (y-u)g''(u)du; x\right)\right\|_1 &\leq C\int_0^\infty \delta_\eta^2|g''(u)|(u)du \\
&= C\delta_\eta^2(u)\int_0^\infty |g''(u)|du \leq \frac{C}{\eta}\|\varphi^2 g''\|_1
\end{aligned}
$$

Hence it is also verified for $p = 1$. Thus Riesz-Thorin theorem yields the complete proof of this theorem for $1 \leq p \leq \infty$. □

Direct Theorem

Theorem 2. *If $r \in L_p[0,\infty)$, $1 \leq p \leq \infty$ and $\varphi(x) = \sqrt{x}$, there holds*

$$\|\chi_\eta(r;x) - r(x)\|_p \leq C\left\{\omega_\varphi^2(r, \frac{1}{\sqrt{\eta}})_p + \omega_1(r, \frac{1}{\eta})_p + \frac{1}{\eta}\|r\|_p\right\}$$

where

$$\omega_1(r,\frac{1}{\eta})_p=\sup_{0<t\leq\frac{1}{\eta}}\|\Delta_y^1 r(x)\|_p$$

and

$$\Delta_y^1 r(x)=r(x+\frac{y}{2})-f(x-\frac{y}{2}).$$

Here ω_1 represents the modulus of continuity of order 1.

Proof. Let us suppose a function $\xi_\eta(r;x)$ such as

$$\xi_\eta(r;x)=\chi_\eta(r;x)+\zeta_\eta(r;x),\ \ \zeta_\eta(r;x)=r(x)-r\left(x+\frac{1}{\eta}\right) \tag{3.1}$$

It gives $\xi_\eta(1;x)=1$, $\xi_\eta(y-x;x)=\frac{2}{\eta}$, $\xi_\eta((y-x)^2;x)\leq\frac{C\delta_\eta^2(x)}{\eta}$ and $\|\xi_\eta\|\leq 3$. Hence by definition

$$\|\zeta_\eta(r;x)\|_p=\left\{\int_0^\infty\left|r(x)-r\left(x+\frac{1}{\eta}\right)\right|^p dx\right\}^{1/p}\leq\omega_1(r,\frac{1}{\eta})_p.$$

By the property of $K-$functional, we have

$$\overline{K}_\varphi^2(r,t^2)=\inf_{g\in\overline{D}}\{\|r-g\|_p+y^2\|\varphi^2g''\|_p+y^4\|g''\|_p\}$$

where $\overline{D}=\{g\in Lp[0,\infty):g'\in A.C._{loc};g'',\varphi^2g''\in L_p[0,\infty)\}$. Hence

$$\|r-g\|_p+\frac{1}{\eta}\|\varphi^2g''\|_p+\frac{1}{\eta^2}\|g''\|_p\leq 2\overline{K}_\varphi^2\left(r,\frac{1}{\eta}\right). \tag{3.2}$$

Applying Taylor formula with remainder as integral, we get

$$\begin{aligned}
&\|\xi_\eta(g;x)-g(x)\|_p\\
\leq\ &\|g'(x)\xi_\eta(y-x;x)\|_p+\left\|\xi_\eta\left(\int_x^y(y-u)g''(u)du;x\right)\right\|_p\\
\leq\ &\frac{2}{\eta}\|g'(x)\|_p+\left\|\chi_\eta\left(\int_x^y(y-u)g''(u)du;x\right)\right\|_p\\
&+\left\|\int_x^{x+\frac{1}{\eta}}(x+\frac{1}{\eta}-u)g''(u)du\right\|_p
\end{aligned}$$

Recalling $\left\|\int_x^{x+\frac{1}{\eta}}\left(x+\frac{1}{\eta}-u\right)g''du\right\|_p\leq\frac{1}{\eta^2}\|g''\|_p$, Lemma 8 and taking $\|g'\|_p\leq\|g''\|_p+\frac{2}{\eta}\|g\|_p$, we obtain

$$\|\xi_\eta(g;x)-g(x)\|_p\leq C\overline{K}_\varphi^2\left(r,\frac{1}{\eta}\right)+\frac{2}{\eta}\|g\|_p \tag{3.3}$$

Combining (3.1)-(3.3), we get the required result as

$$\begin{aligned}\|\chi_\eta(g;x)-g(x)\|_p &\leq \|\xi_\eta(g;x)-g(x)\|_p+\|\zeta_\eta(r;x)\|_p\\ &\leq 4\|r-g\|_p+\|\xi_\eta g-g\|_p+\omega_1\left(r,\frac{1}{\eta}\right)_p\\ &\leq C\left\{\omega_\varphi^2\left(r,\frac{1}{\sqrt{\eta}}\right)_p+\omega_1\left(r,\frac{1}{\eta}\right)_p+\frac{1}{\eta}\|r\|_p\right\}.\end{aligned}$$

□

4 Conclusion

By all counts and with proven results, it is no wonder to say that our operators considered in this research article are very compatible to the discipline of approximation theory. Results and proof of main theorem are very precisely explained. Eventually, we may conclude that this research paper is explicit.

References

[1] DeVore R.A. and Lorentz G.G., *Constructive Approximation*, Springer, Berlin (1993).

[2] Ditizan, Z. and Totik, V., *Moduli of Smoothness [M],* Springer-Verlag, New York, 1987.

[3] Ditizan, Z. and Totik, V., *$K-$functionals and weighted moduli of smoothness [J],* J. Approx. Theory. 63(1) (1990), 3–29.

[4] Garg, S., *Simultaneous approximation estimates for Beta-Szasz Mirakyan operators,* Int. J. of Sci. and Engg. Res. 8.8 (2017), 622-631.

[5] Gupta V. and Agrawal D., *Approximation results by certain genuine operators of integral type,* Kragujevac J. Math. 42(3)(2018), 335–348.

[6] Gupta, V. and Srivastava, G.S., *On convergence of derivative by Szasz-Mirakyan Baskakov type operators,* Math. Student-India. 64(1) (1995), 195-205.

[7] Karsli H. and Ibikli E., *Convergence rate of a new Bezier variant of Chlodowsky operators to bounded variation functions,* J. Comp. Appl. Math. 212(2) (2008), 431-443.

[8] Sharma, R.R. and Garg, S., *Simultaneous approximation for Beta Baskakov Stancu operators,* Int. J. Innov. Tech. and Explor. Engg. (Scopus). 9.4 (2020), 522-528.

[9] Sharma, P.M. and Garg, S., *Higher order iterates of Szasz Mirakyan Baskakov operators,* Studia Univ. Babes-Bolyai Math. 59.1 (2014)69-76.