Analytical Approach to White-noise Functional Solutions for the Wick-Type Stochastic Fractional Zhiber-Shabat Equation

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Abstract

The main aim of this paper is to give analytical approach to exact solutions for the Wick-type stochastic fractional Zhiber -Shabat equation. By means of Hermite transform, white noise theory and fractional Riccati equation method, white noise functional solutions for the Wick-type stochastic fractional Zhiber -Shabat equation are derived. Exact traveling wave solutions for the variable coefficients space-time fractional Zhiber -Shabat equations are given by using the fractional Riccati equation method. The obtained results include soliton-like, periodic and rational solutions.
Keywords: KdV equations; White noise; Hermite transform; Fractional Riccati equation method.

1 Introduction

This paper is devoted to explore the white noise functional solutions for the variable coefficients Wick-type stochastic fractional Zhiber -Shabat equation as the following form:

$$D_x^{\gamma} U_t + P(t) \diamond e^{\diamond U} + Q(t) \diamond e^{\diamond(-U)} + R(t) \diamond e^{\diamond(-2U)} = 0, \quad 0 < \gamma \leq 1, \tag{1.1}$$

where $D_x^\gamma U$ are the modified Riemann-Liouville derivatives defined by Jumarie[1]

$$D_x^\gamma f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\gamma)} \int_0^x (x-\theta)^{-\gamma-1} [f(\theta) - f(0)] d\theta, & \gamma < 0, \\ \dfrac{1}{\Gamma(1-\gamma)} \dfrac{d}{dx} \int_0^x (x-\theta)^{-\gamma} [f(\theta) - f(0)] d\theta, & 0 < \gamma < 1, \\ \left[f^{(\gamma-n)}(x) \right]^{(n)}, & n \leq \gamma < n+1, \ \ n \in \mathbb{N} \end{cases} \tag{1.2}$$

The coefficients $P(t)$, $Q(t)$ and $R(t)$ are Gaussian white noise functions, and "$\diamond$" is the Wick product on the Kondratiev distribution space$(\mathcal{S})_{-1}$which was defined in [2]. Eq.(1.1) can be considered as the Wick version of the following variable coefficients fractional KdV equation:

$$\frac{d^\alpha}{dx^\alpha} u_t + p(t)e^u + q(t)e^{-u} + r(t)e^{-2u} = 0, \ \ (x,t) \in \mathbb{R} \times \mathbb{R}_+, \ \ 0 < \alpha \leq 1, \tag{1.3}$$

where$p(t)$, $q(t)$and$r(t)$are bounded measurable or integrable functions on$\mathbb{R}_+$, this means that Eq.(1.1) can be regarded as the perturbation of Eq.(1.3). El Wakil et al[3] asserted that, Eq.(1.3) is the mathematical model for small but finite amplitude electron-acoustic solitary waves in plasma of cold electron fluid with two different temperature isothermal ions. Therefore, if this model is perturbed by Gaussian white noise, Eq.(1.1) regarded as the mathematical model for the resultant phenomenon.

Since Wadati first introduced and studied stochastic KdV equations [4], many authors, e.g., Xie [5-6], Chen [7-8], Ghany [9], Ghany et al [10-12] and so on, have investigated more intensively the stochastic partial differential equations. In the past several decades, many authors mainly had paid attention to study the nonlinear fractional partial differential equations and gave approximative solutions by using various methods, among these are homotopy perturbation method [13-14], variational iteration method [15], Adomian's decomposition method [16-17] etc.

In the present paper, with the help of Hermite transform, inverse Hermite transform, white noise theory, fractional sub-equation method and fractional Riccati equation method, I will give white noise functional solutions for the Wick-type stochastic space fractional Zihber-Shabat equation and new family of exact traveling wave solutions for the variable coefficients space fractional

Zihber-Shabat equations. New family of exact analytical solutions for the space-time fractional Zihber-Shabat equations with the modified Riemann-Liouville derivative. The obtained results include generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solutions.

2 Modified fractional derivative on the space $(\mathcal{S})_{-1}$

Suppose that $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are the Hida test function space and the Hida distribution space on $\mathbb{R}^d$, respectively. Let $h_n(x)$ be Hermite polynomials and put

$$\zeta_n = e^{-x^2} h_n(\sqrt{2}x)/((n-1)!\pi)^{\frac{1}{2}}, \quad n \geqslant 1. \tag{2.1}$$

then, the collection $\{\zeta_n\}_{n\geqslant 1}$ constitutes an orthogonal basis for $L_2(\mathbb{R})$.
Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ denote d-dimensional multi-indices with $\alpha_1, \alpha_2, ..., \alpha_d \in \mathbb{N}$. The family of tensor products

$$\zeta_\alpha := \zeta_{(\alpha_1,\alpha_2,...,\alpha_d)} = \zeta_{\alpha_1} \otimes \zeta_{\alpha_2} \otimes ... \otimes \zeta_{\alpha_d} \tag{2.2}$$

forms an orthogonal basis for $L_2(\mathbb{R}^d)$.
Suppose that $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, ..., \alpha_d^{(i)})$ is the i-th multi-index number in some fixed ordering of all d-dimensional multi-indices α. We can, and will, assume that this ordering has the property that

$$i < j \Rightarrow \alpha_1^{(i)} + \alpha_2^{(i)} + ... + \alpha_d^{(i)} < \alpha_1^{(j)} + \alpha_2^{(j)} + ... + \alpha_d^{(j)} \tag{2.3}$$

i.e., the $\{\alpha^{(j)}\}_{j=1}^{\infty}$ occurs in an increasing order. Now
Define

$$\eta_i := \zeta_{\alpha_1^{(i)}} \otimes \zeta_{\alpha_2^{(i)}} \otimes ... \otimes \zeta_{\alpha_d^{(i)}}, \quad i \geqslant 1. \tag{2.4}$$

We need to consider multi-indices of arbitrary length. For simplification of notation, we regard multi-indices as elements of the space $(\mathbb{N}_0^{\mathbb{N}})_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ with elements $\alpha_i \in \mathbb{N}_0$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. We write $J = (\mathbb{N}_0^{\mathbb{N}})_c$, for $\alpha \in J$, Define

$$H_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}(< \omega, \eta_i >), \quad \omega = (\omega_1, \omega_2, ..., \omega_d) \in S'(\mathbb{R}^d) \tag{2.5}$$

For a fixed $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, suppose the space $(S)_1^n$ consists of those $f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in \bigoplus_{k=1}^n L_2(\mu)$ with $c_\alpha \in \mathbb{R}^n$ such that

$$||f||_{1,k}^2 = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty \tag{2.6}$$

where, $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n (c_\alpha^{(k)})^2$ if $c_\alpha = (c_\alpha^{(1)}, c_\alpha^{(2)}, ..., c_\alpha^{(n)}) \in \mathbb{R}^n$ and μ is the white noise measure on $(S'(\mathbb{R}), B(S'(\mathbb{R})))$, $\alpha! = \prod_{k=1}^\infty \alpha_k!$ and $(2\mathbb{N})^\alpha = \prod_j (2j)^{\alpha_j}$ for $\alpha \in J$.

The space $(S)_{-1}^n$ consists of all formal expansions $F(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega)$ with $b_\alpha \in \mathbb{R}^n$ such that $||f||_{-1,-q} = \sum_\alpha b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$ for some $q \in \mathbb{N}$. The family of seminorms $||f||_{1,k}, k \in \mathbb{N}$ gives rise to a topology on $(S)_1^n$, and we can regard $(S)_{-1}^n$ as the dual of $(S)_1^n$ by the action

$$< F, f >= \sum_\alpha (b_\alpha, c_\alpha)\alpha! \tag{2.7}$$

where (b_α, c_α) is the inner product in $\mathbb{R}^n$.

The Wick product $f \diamond F$ of two elements $f = \sum_\alpha a_\alpha H_\alpha, F = \sum_\beta b_\beta H_\beta \in (S)_{-1}^n$ with $a_\alpha, b_\beta \in \mathbb{R}^n$, is defined by

$$f \diamond F = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta} \tag{2.8}$$

The spaces $(S)_1^n, (S)_{-1}^n, S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are closed under Wick products.

For $F = \sum_\alpha b_\alpha H_\alpha \in (S)_{-1}^n$, with $b_\alpha \in \mathbb{R}^n$, the Hermite transformation of F, is defined by

$$\mathcal{H}F(z) = \widetilde{F}(z) = \sum_\alpha b_\alpha z^\alpha \in \mathbb{C}^N \tag{2.9}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^N$ (the set of all sequences of complex numbers) and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$, if $\alpha \in J$, where $z_j^0 = 1$.

For $F, G \in (S)_{-1}^n$ we have

$$\widetilde{F \diamond G}(z) = \widetilde{F}(z).\widetilde{G}(z) \tag{2.10}$$

for all z such that $\widetilde{F}(z)$ and $\widetilde{G}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of $\mathbb{C}^N$ defined by $(z_1^1, z_2^1, ..., z_n^1).(z_1^2, z_2^2, ..., z_n^2) = \sum_{k=1}^n z_k^1 z_k^2$.

Let $X = \sum_\alpha a_\alpha H_\alpha$, then the vector $c_0 = \widetilde{X}(0) \in \mathbb{R}^N$ is called the generalized expectation of X which denoted by $E(X)$. Suppose that $g : U \longrightarrow \mathbb{C}^M$ is an analytic function, where U is a neighborhood of $E(X)$. Assume that the Taylor series of g around $E(X)$ have coefficients in $\mathbb{R}^M$. Then the Wick version $g^\diamond(X) = \mathcal{H}^{-1}(g \circ \widetilde{X}) \in (S)_{-1}^M$. In other words, if g has the power series expansion $g(z) = \sum a_\alpha (z - E(X))^\alpha$, with $a_\alpha \in \mathbb{R}^M$, then $g^\diamond(z) = \sum a_\alpha (z - E(X))^{\diamond\alpha} \in (S)_{-1}^M$.

Suppose that modelling consideration leads us to consider an stochastic fractional PDE as follows:

$$A(t, x, \partial_{t^\alpha}, \Delta_{x^\beta}, U, \omega) = 0 \tag{2.11}$$

where A is some given function $U = U(t, x, \omega)$ is an unknown (generalized) stochastic process, and the operators $\partial_{t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}, \Delta_{x^\beta} = (\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}, \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}}, ..., \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}})$ when $x = (x_1, x_2, ..., x_d), \beta = (\beta_1, \beta_2, ..., \beta_d) \in \mathbb{R}^d$. Firstly, we interpret all products as Wick products and all functions as their Wick versions. Wick version of Eq. (2.11) is written as follows:

$$A^\diamond(t, x, \partial_{t^\alpha}, \Delta_{x^\beta}, U, \omega) = 0 \tag{2.12}$$

Secondly, we take the Hermite transformation of Eq. (2.12), which turns Wick products into ordinary products (between complex numbers), so the equation takes the form

$$\widetilde{A}(t, x, \partial_{t^\alpha}, \Delta_{x^\beta}, \widetilde{U}, z_1, z_2, ...) = 0 \tag{2.13}$$

where $\widetilde{U} = \mathcal{H}(U)$ is the Hermite transformation of U and $z_1, z_2, ...$ are complex numbers.

Definition 2.1. A measurable function

$$u : \mathbb{R}^d \longrightarrow (S)_{-1}^N$$

is called $(S)_{-1}^N$-process. The partial derivative $\frac{\partial u}{\partial x_k}$ of an $(S)_{-1}^N$ u is defined by

$$\frac{\partial u}{\partial x_k}(x_1,...,x_d) = lim_{\Delta x_k \to 0} \frac{u(x_1,...x_k+\Delta x_k, x_d) - u(x_1,...,x_d)}{\Delta x_k}$$

provided the limit exists in $(S)_{-1}^N$. Let$u$be a continuous$(\mathcal{S})_{-1}$-process, and let$h > 0$denote a constant discretization span. Define the forward operator$FW_{x_k}(h)$,by

$$FW_{x_k}(h)u(x) := u(x_1,...,x_k+h, x_{k+1},...,x_d). \tag{2.14}$$

Then for$0 < \alpha \leq 1$,theα-orderfractional difference of$u$is defined by the expression

$$\Delta_{x_k}^{\alpha} u(x) := (FW_{x_k}(h) - 1)^{\alpha}.u(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} u(x_1,...,x_k+(\alpha-j)h, x_{k+1},...,x_d), \tag{2.15}$$

and itsα-orderfractional derivative is given by

$$D_{x_k}^{\alpha} u(x) = \lim_{h \downarrow 0} \frac{\Delta_{x_k}^{\alpha} u(x)}{h^{\alpha}}. \tag{2.16}$$

provided the limit exists in$(\mathcal{S})_{-1}$.

In terms of the Hermite transform the limit on the right-hand side of (2.14) exists if and only if there exists an element$Y \in (\mathcal{S})_{-1}$such that$\frac{1}{h^{\alpha}} \Delta_{x_k}^{\alpha} \widetilde{u}(x,z) \to \widetilde{Y}(z)$pointwise boundedly (uniformly) in$K_{\sigma}(\delta)$for some$\sigma < \infty, \delta > 0$,where

$$K_{\sigma}(\delta) = \left\{ z = (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}} : \sum_{\mu \neq 0} |z^{\mu}|^2 (2\mathbb{N})^{\sigma\mu} < \delta \right\}$$

If this is the case, then$Y$is denoted by$D_{x_k}^{\alpha} u(x)$.

Let us denote by$L^1(a,b;(\mathcal{S})_{-1})$the space of all strongly integrable$(\mathcal{S})_{-1}$-processeson$[a,b]$,then for$X \in L^1(a,b;(\mathcal{S})_{-1})$we can set the$\alpha$-orderRiemann-Liouville fractional integral operator and the modified Riemann-Liouville fractional derivative as follows:

Definition 2.2. Theα-orderRiemann-Liouville fractional integral operator of$X$is defined as

$$J^{\alpha}X(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}X(\tau)d\tau,$$
$$\text{for } \alpha > 0,\ t \in [a,b] \text{ and } J^0X(t) := X(t). \tag{2.17}$$

When we apply Hermite transform to solve stochastic differential equations the following observation is important.

Assume that the$(\mathcal{S})_{-1}$-process $X(t,\omega)$has anα-orderfractional derivative and

$$D_t^{\alpha}X(t,\omega) = F(t,\omega) \text{ in } (\mathcal{S})_{-1}, \tag{2.18}$$

this equivalent to saying that

$$\lim_{h\downarrow 0}\frac{\Delta_{x_k}^{\alpha}\widetilde{X}(t,z)}{h^{\alpha}} = \widetilde{F}(t,z) \tag{2.19}$$

uniformly for$z \in K_{\sigma}(\delta)$for some$\sigma < \infty, \delta > 0$.For this it is clearly necessary that

$$D_t^{\alpha}\widetilde{X}(t,z) = \widetilde{F}(t,z) \quad \text{for all } z \in K_{\sigma}(\delta), \tag{2.20}$$

but apparently not sufficient, because we also need that the pointwise convergence is bounded for$z \in K_{\sigma}(\delta)$. The following result is sufficient for our purposes.

Lemma 2.1. *Suppose$X(t,\omega)$and$F(t,\omega)$are$(\mathcal{S})_{-1}$-processessuch that*

(*i*) $D_t^{\alpha}\widetilde{X}(t,z) = \widetilde{F}(t,z)$*for each*$(t,z) \in (a,b)\times K_{\sigma}(\delta)$*and that*

(*ii*) $\widetilde{F}(t,z)$*is a bounded function for*$(t,z) \in (a,b)\times K_{\sigma}(\delta)$*and continuous with respect to*$t \in (a,b)$*for each*$z \in K_{\sigma}(\delta)$.

Then$X(t,\omega)$*has an*α*-orderfractional derivative and for each*$t \in (a,b)$

$$D_t^{\alpha}X(t,\omega) = F(t,\omega) \quad \textit{in } (\mathcal{S})_{-1}. \tag{2.21}$$

Proof. According to the fractional counterpart of the mean value theorem [**?**], we have

$$\frac{1}{h^{\alpha}}\Delta_t^{\alpha}\widetilde{X}(t,z)=\frac{\Gamma(1+\alpha)}{h^{\alpha}}\left(\widetilde{X}(t+h,z)-\widetilde{X}(t,z)\right)=\widetilde{F}(t+\theta h,z), \tag{2.22}$$

for some$\theta\in[0,1]$and for each$z\in K_{\sigma}(\delta)$.So if the hypotheses (i), (ii) hold, then

$$\lim_{h\downarrow 0}\frac{\Delta_t^{\alpha}\widetilde{X}(t,z)}{h^{\alpha}}=\widetilde{F}(t,z) \tag{2.23}$$

pointwise boundedly for$z\in K_{\sigma}(\delta)$. ■

Taking Hermite transform of (2.15) and using [2, Lemma 2.8.5], we get the following result

Lemma 2.2. *Let$X(t)$be an$(\mathcal{S})_{-1}$-process.Suppose there exist$\sigma<\infty$,$\delta>0$such that*

$$\sup\{\widetilde{X}(t,z):t\in[a,b],\ z\in K_{\sigma}(\delta)\}<\infty \tag{2.24}$$

and$\widetilde{X}(t,z)$is a continuous function with respect to$t\in[a,b]$for each$z\in K_{\sigma}(\delta)$.Then theα-order Riemann-Liouville fractional integral operator of$X(t)$exists and

$$\widetilde{J^{\alpha}X(t)}(z)=J^{\alpha}\widetilde{X}(t,z),\quad for\ \alpha\geq 0,\ t\in[a,b],\ z\in K_{\sigma}(\delta). \tag{2.25}$$

In the case of higher order derivatives we have the following result

Lemma 2.3. *Suppose there exist an open interval$I$,real numbers$\sigma,\delta$and a function$u:I\times K_{\sigma}(\delta)\to\mathbb{C}$such that*

$$D_x^{2\alpha}u(x,z)=\widetilde{F}(x,z),\ \ for\ (x,z)\in I\times K_{\sigma}(\delta) \tag{2.26}$$

where$F(x)\in(\mathcal{S})_{-1}$for all$x\in I$. Suppose$D_x^{2\alpha}u$is bounded for$(x,z)\in I\times K_{\sigma}(\delta)$and continuous with respect to$x\in I$for each$z\in K_{\sigma}(\delta)$.Then there exists$U(x)\in(\mathcal{S})_{-1}$such that

$$D_x^{2\alpha}U(x)=F(x),\ \ for\ x\in I. \tag{2.27}$$

Proof. By the fractional counterpart of the mean value theorem again, we have

$$\frac{1}{h^{2\alpha}}\Delta_x^{2\alpha}u(x,z)=\frac{\Gamma^2(1+\alpha)}{h^{2\alpha}}\big(u(x+2h,z)-2u(x+h,z)+u(x,z)\big)=\widetilde{F}(x+\theta h,z) \tag{2.28}$$

for some$\theta\in[0,1]$and for each$z\in K_\sigma(\delta)$. So if (2.14) and the assumptions on$D_x^{2\alpha}u$hold, then

$$\lim_{h\downarrow 0}\frac{\Delta_x^{2\alpha}\widetilde{u}(t,z)}{h^{2\alpha}}=\widetilde{F}(x,z) \tag{2.29}$$

pointwise boundedly for$z\in K_\sigma(\delta)$. According to [2, Lemma 2.8.5], we can apply the inverse Hermite transform to Eq.(2.17) and get

$$D_x^{2\alpha}U(x)=F(x)\ \text{ in } (\mathcal{S})_{-1} \text{ and for all } x\in I, \tag{2.30}$$

where$u(x,z)=\widetilde{U}(x)(z)$for all$(x,z)\in I\times K_\sigma(\delta)$ ■

More generally , we can apply the argument of Lemma 2.1 repeatedly and get the following result

Theorem 2.4. *Suppose$u(x,t,z)$is a solution (in the usual strong, pointwise sense) of the equation*

$$\widetilde{\Omega}(x,t,D_t^\alpha,D_{x_1}^\alpha,...,D_{x_d}^\alpha,u,z)=0 \tag{2.31}$$

for(x,t)in some bounded open set$G\subset\mathbb{R}^d\times\mathbb{R}_+$,and for all$z\in K_\sigma(\delta)$,for some$\sigma,\delta$. Moreover, suppose that$u(x,t,z)$and all its partial fractional derivatives, which are involved in (2.19), are (uniformly) bounded for$(x,t,z)\in G\times K_\sigma(\delta)$,continuous with respect to$(x,t)\in G$for each$z\in K_\sigma(\delta)$and analytic with respect to$z\in K_\sigma(\delta)$,for all$(x,t)\in G$.Then there exists$U(x,t)\in(\mathcal{S})_{-1}$ such that$u(x,t,z)=\widetilde{U}(t,x)(z)$for all$(t,x,z)\in G\times K_\sigma(\delta)andU(x,t)$solves (in the strong sense) the equation

$$\Omega^\diamond(t,x,D_t^\alpha,D_{x_1}^\alpha,...,D_{x_d}^\alpha,U,\omega)=0\quad in\ \ (\mathcal{S})_{-1}. \tag{2.32}$$

3 White Noise Functional Solutions of Eq.(1.1)

Taking the Hermite transform of Eq.(1.1), we get the deterministic equation:

$$D_x^{\gamma}\widetilde{U}_t(x,t,z)+\widetilde{P}(t,z)e^{\widetilde{U}(x,t,z)}+\widetilde{Q}(t,z)e^{-\widetilde{U}(x,t,z)}+\widetilde{R}(t,z)e^{-2\widetilde{U}(x,t,z)}=0,\ \ 0<\gamma\leq 1, \tag{3.1}$$

where$z=(z_1,z_2,...)\in(\mathbb{C}^{\mathbb{N}})_c$is a vector parameter. For the sake of simplicity we denote$P(t,z)=\widetilde{P}(t,z)$,$Q(t,z)=\widetilde{Q}(t,z)$,$R(t,z)=\widetilde{R}(t,z)andu(x,t,z)=\widetilde{U}(x,t,z)$. To determine the solution $u(x,t,z)$ explicitly, we first introduce the following transformations:

$$u=u(\xi),\quad \xi=f(t,z)x+g(t,z), \tag{3.2}$$

where$f(t,z)$and$g(t,z)$are functions to be determined later, then Eq.(3.1) is reduced into a fractional ordinary differential equation:

$$f_t^{\gamma}D_{\xi}^{\gamma}u(\xi)+P(t,z)e^{u(\xi)}+Q(t,z)e^{-u(\xi)}+R(t,z)e^{-2u(\xi)}=0,\ \ 0<\gamma\leq 1, \tag{3.3}$$

We next suppose that Eq.(3.3) has a solution in the form:

$$u=\sum_{i=0}^{n}a_i(t,z)\ F^i(\xi), \tag{3.4}$$

where$a_i\ (i=0,1,...,n)$are functions to be determined later,$n$is a positive integer and$F$satisfies the fractional Riccati equation:

$$D_{\xi}^{\gamma}F=\sigma+F^2, \tag{3.5}$$

where$\sigma$is an arbitrary constant. Balancing$uD_{\xi}^{\beta}u$with$D_{\xi}^{3\beta}u$in Eq.(3.3) gives$n=2$. So (3.4) can be simplified as following:

$$u=a_0+a_1F+a_2F^2 \tag{3.6}$$

By substituting Eq.(3.6) along with Eq.(3.5) into Eq.(3.3) and collect the coefficients of$F^i(i=0,1,...,5)$and set them to be zero, we will obtain the following set of algebraic equations in the unknowns$a_i(i=0,1,2), f$ and$g$:

$$\begin{cases} \sigma_\alpha a_1(f_t x+g_t)^\alpha+\sigma_\beta a_0 a_1 f^\beta p+2\sigma_\beta^2 a_1 f^{3\beta} q=0, \\ 2\sigma_\alpha a_2(f_t x+g_t)^\alpha+\sigma_\beta(2a_0a_2+a_1^2) f^\beta p+16\sigma_\beta^2 a_2 f^{3\beta} q=0, \\ a_1(f_t x+g_t)^\alpha+(a_0a_1+3\sigma_\beta a_1a_2) f^\beta p+8\sigma_\beta a_1 f^{3\beta} q=0, \\ 2a_2(f_t x+g_t)^\alpha+(2a_0a_2+a_1^2+2\sigma_\beta a_2^2) f^\beta p+40\sigma_\beta a_2 f^{3\beta} q=0, \\ 3a_1a_2 f^\beta p+6a_1 f^{3\beta} q=0, \\ 2a_2^2 f^\beta p+24a_2 f^{3\beta} q=0. \end{cases} \tag{3.7}$$

With aid of the symbolic computation system *Maple*, we can find the following sets of solutions of the system (3.7):

$$\begin{aligned} a_0=\lambda,\ \ a_1=0,\ \ a_2=-12k^{2\beta}\frac{q}{p},\ \ f(t,z)=k, \\ g(t,z)=-\frac{k\sigma_\beta}{\sigma_\alpha}\int_0^t(\lambda p(s,z)+8\sigma_\beta k^{2\beta} q(s,z))^{\frac{1}{\alpha}}ds, \end{aligned} \tag{3.8}$$

where$\lambda$and$k$are arbitrary constants.

In a recent paper by Zhang et al. [18-19], a set of five different solutions to Eq.(3.5) was introduced as follows:

$$F(\xi)=\begin{cases} -\sqrt{-\sigma_\gamma}\tanh_\gamma(-\sqrt{-\sigma_\gamma}\xi), & \sigma_\gamma<0, \\ -\sqrt{-\sigma_\gamma}\coth_\gamma(-\sqrt{-\sigma_\gamma}\xi), & \sigma_\gamma<0, \\ \sqrt{\sigma_\gamma}\tan_\gamma(\sqrt{\sigma_\gamma}\xi), & \sigma_\gamma>0, \\ \sqrt{\sigma_\gamma}\cot_\gamma(\sqrt{\sigma_\gamma}\xi), & \sigma_\gamma>0, \\ -\dfrac{\Gamma(1+\gamma)}{\xi^\gamma+\omega}, & \omega=\text{const.},\ \ \sigma_\gamma=0, \end{cases} \tag{3.9}$$

where the generalized hyperbolic and trigonometric functions [20] are expressed by the following:

$$\tanh_\gamma(x)=\frac{E_\gamma(x^\gamma)-E_\gamma(-x^\gamma)}{E_\gamma(x^\gamma)+E_\gamma(-x^\gamma)},\quad \coth_\gamma(x)=\frac{E_\gamma(x^\gamma)+E_\gamma(-x^\gamma)}{E_\gamma(x^\gamma)-E_\gamma(-x^\gamma)},$$

$$\tan_\gamma(x) = -i\,\frac{E_\gamma(ix^\gamma) - E_\gamma(-ix^\gamma)}{E_\gamma(ix^\gamma) + E_\gamma(-ix^\gamma)}, \quad \cot_\gamma(x) = i\,\frac{E_\gamma(ix^\gamma) + E_\gamma(-ix^\gamma)}{E_\gamma(ix^\gamma) - E_\gamma(-ix^\gamma)},$$

where$E_\gamma$denotes the Mittag-Leffler function [20], defined by:

$$E_\gamma(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(j\gamma+1)}.$$

We therefore obtain from Eqs.(3.9), (3.2), (3.6) and (3.8) three types of exact solitary wave solutions of Eq.(3.1), namely:

- Four generalized hyperbolic function solutions ($\sigma_\gamma < 0$):

$$u_1 = \lambda + 12\sigma_\gamma k^{2\beta}\frac{q}{p}\tanh_\gamma^2\left(-\sqrt{-\sigma_\gamma}kx + \sqrt{-\sigma_\gamma}k\int_0^t (\lambda p(s,z) + 8\sigma_\beta k^{2\beta} q(s,z))^{\frac{1}{\alpha}}ds\right) \quad (3.10)$$

$$u_2 = \lambda + 12\sigma_\gamma k^{2\beta}\frac{q}{p}\coth_\gamma^2\left(-\sqrt{-\sigma_\gamma}kx + \sqrt{-\sigma_\gamma}k\int_0^t (\lambda p(s,z) + 8\sigma_\beta k^{2\beta} q(s,z))^{\frac{1}{\alpha}}ds\right) \quad (3.11)$$

- Four generalized trigonometric function solutions ($\sigma_\gamma > 0$):

$$u_3 = \lambda - 12\sigma_\gamma k^{2\beta}\frac{q}{p}\tan_\gamma^2\left(\sqrt{\sigma_\gamma}kx + \sqrt{\sigma_\gamma}k\int_0^t (\lambda p(s,z) + 8\sigma_\beta k^{2\beta} q(s,z))^{\frac{1}{\alpha}}ds\right) \quad (3.12)$$

$$u_4 = \lambda - 12\sigma_\gamma k^{2\beta}\frac{q}{p}\cot_\gamma^2\left(\sqrt{\sigma_\gamma}kx + \sqrt{\sigma_\gamma}k\int_0^t (\lambda p(s,z) + 8\sigma_\beta k^{2\beta} q(s,z))^{\frac{1}{\alpha}}ds\right) \quad (3.13)$$

- One rational solution ($\sigma_\gamma = 0$):

$$u_5 = \lambda - \frac{12k^{2\beta}\Gamma^2(1+\gamma)q}{p\left(\left(kx - k(\lambda)^{\frac{1}{\alpha}}\int_0^t (p(s,z))^{\frac{1}{\alpha}}ds\right)^\gamma + \omega\right)^2}. \quad (3.14)$$

Recalling the result stated in Theorem 2.4., and by virtue of Lemma 2.1, we know that there exists$U(x,t) \in (\mathcal{S})_{-1}$such that$u(x,t,z) = \widetilde{U}(x,t)(z)$for all$(x,t,z) \in G \times K_r(q)$, where$U(x,t)$is the inverse Hermite transform of$u(x,t,z)$. Consequently, $U(x,t)$solves Eq.(1.1). Hence, for$P(t) \neq$

0the white noise functional solutions of Eq.(1.1) are as follows:

$$U_1(x,t) = \lambda + 12\sigma_\gamma k^{2\beta}\frac{Q(t)}{P(t)}\tanh_\gamma^{\diamond 2}\left(-\sqrt{-\sigma_\gamma}kx + \sqrt{-\sigma_\gamma}k\int_0^t (\lambda P(s) + 8\sigma_\beta k^{2\beta}Q(s))^{\diamond\alpha^{-1}}ds\right), \quad \sigma_\gamma < 0, \tag{3.15}$$

$$U_2(x,t) = \lambda + 12\sigma_\gamma k^{2\beta}\frac{Q(t)}{P(t)}\coth_\gamma^{\diamond 2}\left(-\sqrt{-\sigma_\gamma}kx + \sqrt{-\sigma_\gamma}k\int_0^t (\lambda P(s) + 8\sigma_\beta k^{2\beta}Q(s))^{\diamond\alpha^{-1}}ds\right), \quad \sigma_\gamma < 0, \tag{3.16}$$

$$U_3(x,t) = \lambda - 12\sigma_\gamma k^{2\beta}\frac{Q(t)}{P(t)}\tan_\gamma^{\diamond 2}\left(\sqrt{\sigma}kx + \sqrt{\sigma}k\int_0^t (\lambda P(s) + 8\sigma_\beta k^{2\beta}Q(s))^{\diamond\alpha^{-1}}ds\right), \quad \sigma_\gamma > 0, \tag{3.17}$$

$$U_4(x,t) = \lambda - 12\sigma_\gamma k^{2\beta}\frac{Q(t)}{P(t)}\cot_\gamma^{\diamond 2}\left(\sqrt{\sigma}kx + \sqrt{\sigma}k\int_0^t (\lambda P(s) + 8\sigma_\beta k^{2\beta}Q(s))^{\diamond\alpha^{-1}}ds\right), \quad \sigma_\gamma > 0, \tag{3.18}$$

$$U_5(x,t) = \lambda - \frac{12k^{2\beta}\Gamma^2(1+\gamma)Q(t)}{P(t)\left(\left(kx - k(\lambda)^{\frac{1}{\alpha}}\int_0^t (P(s))^{\diamond\gamma^{-1}}ds\right)^{\diamond\alpha} + \omega\right)^{\diamond 2}}, \quad \sigma_\gamma = 0. \tag{3.19}$$

4 Conclusion

In this paper, Hermite transform, white noise theory and Fractional Riccati equation method are applied successfully for constructing some white noise functional solutions for the Wick-type stochastic fractional KdV equations and a new family of exact analytical solutions for the fractional KdV equations with the modified Riemann-Liouville derivative. The obtained results include generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solutions. The method which we have proposed in this paper can be used for solving other nonlinear stochastic fractional partial differential equations with nonlinear terms of any order. Also, we have only discussed the solutions of stochastic fractional KdV equations driven by Gaussian white noise. There is a unitary mapping between the Gaussian white noise space and the Poisson white noise space, this connection was given by Benth and Gjerde [21]. Hence, with the help of this connection, we can derive some Poisson white noise functional solutions, if the coefficients$P(t)$, $Q(t)$are Poisson white noise functions in Eq.(1.1). We note that as $\alpha \to 1$, all the obtained results give a new set of exact analytical solutions for the well known Wick-type stochastic KdV equations. All solutions obtained in this paper have been checked by *Maple* software. Moreover, we observe that we can get different solutions for different forms of the coefficients $P(t)$ and $Q(t)$.

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