Qualitative Behavior and Solutions of Sixth Rational Difference Equations

[1,2]E. M. Elasyed, [1,3]Mai T. Alharthi*

[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Kingdom of Saudi Arabia

[2]Department of Mathematics Faculty of Science, Mansoura University, Mansoura, 35516, EGYPT

[3]Department of Mathematics Faculty of Science, Jeddah University, Jeddah, Saudi Arabia

*Corresponding author: miami-992@hotmail.com

Abstract

In this article, we introduced the solutions of the following difference equations

$$z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(\pm 1 \pm z_{n-4}z_{n-5})}, \quad n = 0, 1, 2, ...,$$

where the initial conditions $z_{-5},\ z_{-4},\ z_{-3},\ z_{-2},\ z_{-1}$ and z_0 are arbitrary non-zero real numbers. Moreover, we presented the solutions of some special cases of these equations and studied the dynamic behavior of the these equations. Finally, we obtained the estimation of the initial coefficients.

Keywords: Difference equation, stability, linearized stability, periodicity.

1 Introduction

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology etc. Difference equations become apparent in the study of

discretization methods for differential equations. Differential equation results have been discovered to produce some results in the theory of difference equations that are more or less natural discrete analogues. For additional results from the research of the rational difference equation see ([1],[61]).
Alayachi et al. [1] interested in studying the global attractivity, boundedness character and the periodic nature of the following nonlinear difference equation

$$z_{n+1} = az_{n-1} + \frac{bz_{n-1}}{cz_{n-1} - dz_{n-3}}.$$

Khalaf-Allah in [40] obtained the formulae of solutions of the difference equations

$$z_{n+1} = \frac{z_{n-2}}{\pm 1 + z_n z_{n-1} z_{n-2}}.$$

Also, he studied the global asymptotic stability of the equilibrium points of these equations via the formulae.
In [36], Gumus and Abo-Zeid determined the forbidden set, introduced an explicit formula for the solutions and discussed the global behavior of solutions of the difference equation

$$z_{n+1} = \frac{az_n z_{n-k+1}}{bz_{n-k+1} + cz_{n-k}}.$$

Okumus and Soykan [48] investigated the stability character, the periodicity and the global behavior of solutions of the following four rational difference equations

$$z_{n+1} = \frac{\pm 1}{z_n(z_{n-1} \pm 1) - 1}, \qquad z_{n+1} = \frac{\pm 1}{z_n(z_{n-1} \pm 1) + 1},$$

In [58], Zhang et al. demonstrated the existence of bounded, asymptotic behavior and the periodicity of the following difference equation

$$z_{n+1} = A + \frac{z_n}{z_{n-1} z_{n-2}}.$$

Elabbasy et al. in [10] studied the qualitative behavior of the solution of the recursive sequence

$$z_{n+1} = a + \frac{dz_{n-l} z_{n-k}}{cz_{n-s} - b}.$$

El-Moneam et al. [13] investigated the local stability, global stability and bounded of solutions of the difference equations

$$z_{n+1} = z_{n-p}\left(\frac{2z_{n-q} + z_{n-r}}{z_{n-q} + z_{n-r}}\right).$$

AbdelKhaliq and Elsayed in [42] studied the solution and periodic of the following difference equations

$$z_{n+1} = \frac{z_{n-1}z_{n-5}}{z_{n-3}(\pm 1 \pm z_{n-1}z_{n-5})}.$$

In this paper, we will investigate the form of solutions and the global asymptotic behavior of the following recursive sequences

$$z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(\pm 1 \pm z_{n-4}z_{n-5})}, \quad n = 0, 1, 2, \ldots .$$

Now, we will introduce some of definitions and theorems that are used in solving the special cases of difference equations:

Definition 1. *Let I be some interval of real numbers and let*

$$F : I^{k+1} \to I,$$

be continuously differentiable function. Then for every set of initial condition $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$,the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2. *A point $x^* \in I$ is called an equilibrium point of Eq.(1.1)if*

$$x^* = F(x^*, x^*, \ldots, x^*),$$

that is, $x_n = x^*$ for $n \geq 0$. is a solution of Eq.(1.1), or equivalently, x^* is a fixed point.

Definition 3. *(Stability)*

Let x^* be an equilibrium point of Eq.(1.1).

1. The equilibrium point x^* of Eq.(1.1), is called **locally stable** if for $\epsilon > 0$, there exist $\delta > 0$ such that for all $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1), and

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \delta,$$

then

$$|x_n - x^*| < \epsilon \quad \text{for all} \quad n \geq 0.$$

2. The equilibrium point x^* of Eq.(1.1), is called **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1), and

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \gamma,$$

then

$$\lim_{n\to\infty} = x^*.$$

3. The equilibrium point x^* of Eq.(1.1) is called **global attractor** if for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1), we have $\lim_{n\to\infty} = x^*$.
4. The equilibrium point x^* of Eq.(1.1), is called **globally asymptotically stable** if it is locally stable and global attractor of Eq.(1.1).
5. The equilibrium point x^* of Eq.(1.1), is called **unstable** if x^* is not locally stable.

2 Linearized Stability Analysis

Definition 4. *The linearized equation of Eq.(1.1), about the equilibrium point x^* is the linear difference equation*

$$y_{n+1} = \sum_{j=0}^{k} \frac{\partial F(x^*, x^*, ..., x^*)}{\partial x_{n-j}} y_{n-j}. \tag{2.1}$$

The characteristic equation associated with Eq.(2.1) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0,$$

where

$$p_j = \frac{\partial F(x^*, x^*, ..., x^*)}{\partial x_{n-j}}.$$

Theorem 1. *[34]*

Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for asymptotic stability of the difference equations

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 1, 2, \dots .$$

Remark 1. *The previous theorem can be extended to a general linear equation of the form*

$$x_{n-k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \qquad n = 0, 1, 2, ..., \tag{2.2}$$

where $p_1, p_{2,}..., p_k \in R$ *and* $k \in \{0, 1, 2, ...\}$, *Then Eq.(2.2) is asymptotically stable if*

$$\sum_{i=1}^{k} |p_i| < 1.$$

3 Qualitative Behavior of Solutions of $z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(1+z_{n-4}z_{n-5})}$

In this section, we study some properties of the following recursive equation in the form

$$z_{n+1}=\frac{z_{n-4}z_{n-5}}{z_n(1+z_{n-4}z_{n-5})}, \tag{3.1}$$

where the initial conditions $z_{-5},\ z_{-4},\ z_{-3},\ z_{-2,}\ z_{-1}$ and z_0 are arbitrary non-zero real numbers.

Theorem 2. *Let* $\{z_n\}_{n=-5}^{\infty}$ *be a solution of difference equation (3.1). Then for* $n = 0, 1, ...,$

$$z_{10n-5} = f\prod_{j=0}^{n-1}\frac{([2jab+1][2jcd+1][2jef+1][(2j+1)bc+1][(2j+1)de+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2jde+1]},$$

$$z_{10n-4} = e\prod_{j=0}^{n-1}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2jde+1])}{[2jab+1][2jcd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]},$$

$$z_{10n-3} = d\prod_{j=0}^{n-1}\frac{[2jab+1][2jcd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2(j+1)de+1]},$$

$$z_{10n-2} = c\prod_{j=0}^{n-1}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2(j+1)de+1]}{[2jab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]},$$

$$z_{10n-1} = b\prod_{j=0}^{n-1}\frac{[2jab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2(j+1)bc+1][2(j+1)de+1]},$$

$$z_{10n} = a\prod_{j=0}^{n-1}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2(j+1)bc+1][2(j+1)de+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]},$$

$$\begin{aligned} z_{10n+1} &= \frac{ef}{a(ef+1)}\prod_{j=0}^{n-1}\frac{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+3)ef+1][2(j+1)bc+1]}\\ &\quad\times\frac{[(2j+1)de+1]}{[[2(j+1)de+1]},\end{aligned}$$

$$\begin{aligned} z_{10n+2} &= \frac{da(ef+1)}{f(de+1)}\prod_{j=0}^{n-1}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+3)ef+1][2(j+1)bc+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1]}\\ &\quad\times\frac{[2(j+1)de+1]}{[(2j+3)de+1]},\end{aligned}$$

$$\begin{aligned} z_{10n+3} &= \frac{cf(de+1)}{a(ef+1)(cd+1)}\prod_{j=0}^{n-1}\frac{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1]}{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1]}\\ &\quad\times\frac{[(2j+1)bc+1][(2j+3)de+1]}{[2(j+1)bc+1][2(j+1)de+1]},\end{aligned}$$

$$\begin{aligned} z_{10n+4} &= \frac{ba(ef+1)(cd+1)}{f(de+1)(bc+1)}\prod_{j=0}^{n-1}\frac{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1]}\\ &\quad\times\frac{[2(j+1)bc+1][2(j+1)de+1]}{[(2j+3)bc+1][(2j+3)de+1]},\end{aligned}$$

where $z_{-5} = f$, $z_{-4} = e$, $z_{-3} = d$, $z_{-2} = c$, $z_{-1} = b$, $z_0 = a$.

Proof. For $n = 0$, the result holds. Now, assume that $n > 0$ and that our assumption holds for $n - 1$. That is

$$z_{10n-15} = f\prod_{j=0}^{n-2}\frac{([2jab+1][2jcd+1][2jef+1][(2j+1)bc+1][(2j+1)de+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2jde+1]},$$

$$z_{10n-14} = e\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2jde+1])}{[2jab+1][2jcd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]},$$

$$z_{10n-13} = d\prod_{j=0}^{n-2}\frac{[2jab+1][2jcd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2(j+1)de+1]},$$

$$z_{10n-12} = c\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2(j+1)de+1]}{[2jab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]},$$

$$z_{10n-11} = b\prod_{j=0}^{n-2}\frac{[2jab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2(j+1)bc+1][2(j+1)de+1]},$$

$$z_{10n-10} = a\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+1)ef+1][2(j+1)bc+1][2(j+1)de+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1][(2j+1)de+1]},$$

$$\begin{aligned} z_{10n-9} &= \frac{ef}{a(ef+1)}\prod_{j=0}^{n-2}\frac{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1]}{[(2j+1)ab+1][(2j+1)cd+1][(2j+3)ef+1][2(j+1)bc+1]} \\ &\times\frac{[(2j+1)de+1]}{[[2(j+1)de+1]}, \end{aligned}$$

$$\begin{aligned} z_{10n-8} &= \frac{da(ef+1)}{f(de+1)}\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+1)cd+1][(2j+3)ef+1][2(j+1)bc+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+1)bc+1]} \\ &\times\frac{[2(j+1)de+1]}{[(2j+3)de+1]}, \end{aligned}$$

$$\begin{aligned} z_{10n-7} &= \frac{cf(de+1)}{a(ef+1)(cd+1)}\prod_{j=0}^{n-2}\frac{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1]}{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1]} \\ &\times\frac{[(2j+1)bc+1][(2j+3)de+1]}{[2(j+1)bc+1][2(j+1)de+1]}, \end{aligned}$$

$$\begin{aligned} z_{10n-6} &= \frac{ba(ef+1)(cd+1)}{f(de+1)(bc+1)}\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1]} \\ &\times\frac{[2(j+1)bc+1][2(j+1)de+1]}{[(2j+3)bc+1][(2j+3)de+1]}, \end{aligned}$$

from Eq. (3.1), we see that

$$\begin{aligned} z_{10n-5} &= \frac{z_{10n-10}z_{10n-11}}{z_{10n-6}(1+z_{10n-10}z_{10n-11})} \\ &= ab\prod_{j=0}^{n-2}\frac{[2jab+1]}{[2(j+1)ab+1]} \\ &\times\frac{1}{\frac{ba(ef+1)(cd+1)}{f(de+1)(bc+1)}\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1][2(j+1)bc+1][2(j+1)de+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+3)bc+1][(2j+3)de+1]}} \\ &\times\frac{1}{(1+ab\prod_{j=0}^{n-2}\frac{[2jab+1]}{[2(j+1)ab+1]})} \end{aligned}$$

$$\begin{aligned} &= \frac{ab[2ab+1][4ab+1]\ldots.[(2n-6)ab+1][(2n-4)ab+1]}{[2ab+1][4ab+1]\ldots[(2n-4)ab+1][(2n-2)ab+1]} \\ &\times\frac{1}{\frac{ba(ef+1)(cd+1)}{f(de+1)(bc+1)}\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1][2(j+1)bc+1][2(j+1)de+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+3)bc+1][(2j+3)de+1]}} \\ &\times\frac{1}{(1+ab\prod_{j=0}^{n-2}\frac{[2jab+1]}{[2(j+1)ab+1]})} \end{aligned}$$

$$\begin{aligned} &= \frac{ab}{[(2n-2)ab+1]} \\ &\times\frac{1}{\frac{ba(ef+1)(cd+1)}{f(de+1)(bc+1)}\prod_{j=0}^{n-2}\frac{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1][2(j+1)bc+1][2(j+1)de+1]}{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+3)bc+1][(2j+3)de+1]}} \\ &\times\frac{1}{(\frac{[(2n-1)ab+1]}{[(2n-2)ab+1]}} \end{aligned}$$

$$\begin{aligned} &= \frac{f(de+1)(bc+1)}{(ef+1)(cd+1)}\times \\ &\times\prod_{j=0}^{n-2}\frac{[2(j+1)ab+1][2(j+1)cd+1][2(j+1)ef+1][(2j+3)bc+1]}{[(2j+1)ab+1][(2j+3)cd+1][(2j+3)ef+1][2(j+1)bc+1]} \\ &\times\frac{[(2j+3)de+1]}{[2(j+1)de+1][(2n-1)ab+1]} \end{aligned}$$

$$
\begin{aligned}
&= f \prod_{j=0}^{n-1} \frac{[(2j+1)bc+1][(2j+1)de+1][2jab+1][2jcd+1][2jef+1]}{[(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2jde+1]} \\
&\quad \times \prod_{j=0}^{n-2} \frac{1}{[(2j+1)ab+1][(2n-1)ab+1]}.
\end{aligned}
$$

$$
= f \prod_{j=0}^{n-1} \frac{(2j+1)bc+1][(2j+1)de+1][2jab+1][2jcd+1][2jef+1]}{[(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2jde+1][(2j+1)ab+1]}.
$$

Consequently, we have

$$
z_{10n-5} = f \prod_{j=0}^{n-1} \frac{[(2j+1)bc+1][(2j+1)de+1][2jab+1][2jcd+1][2jef+1]}{[(2j+1)cd+1][(2j+1)ef+1][2jbc+1][2jde+1][(2j+1)ab+1]}.
$$

Similarly, the other relations can be proved. The proof is completed. □

Theorem 3. *The equilibrium point of Eq.(3.1) is* 0 *and it is not locally asymptotically stable.*

Proof. For the equilibrium points of Eq.(3.1), we can write

$$
z^* = \frac{{z^*}^2}{z^*(1+{z^*}^2)},
$$

then

$$
1 + {z^*}^2 = 1,
$$

thus,

$$
{z^*}^2 = 0.
$$

Then $z^* = 0$ is the unique equilibrium point.

Define the function F by

$$
F(x, y, w) = \frac{yw}{x(1+yw)}.
$$

Then it follows that

$$
\begin{aligned}
F_x(x, y, w) &= \frac{-yw}{x^2(1+yw)}, \ F_y(x, y, w) = \frac{w}{x(1+yw)^2}, \\
F_w(x, y, w) &= \frac{y}{x(1+yw)^2},
\end{aligned}
$$

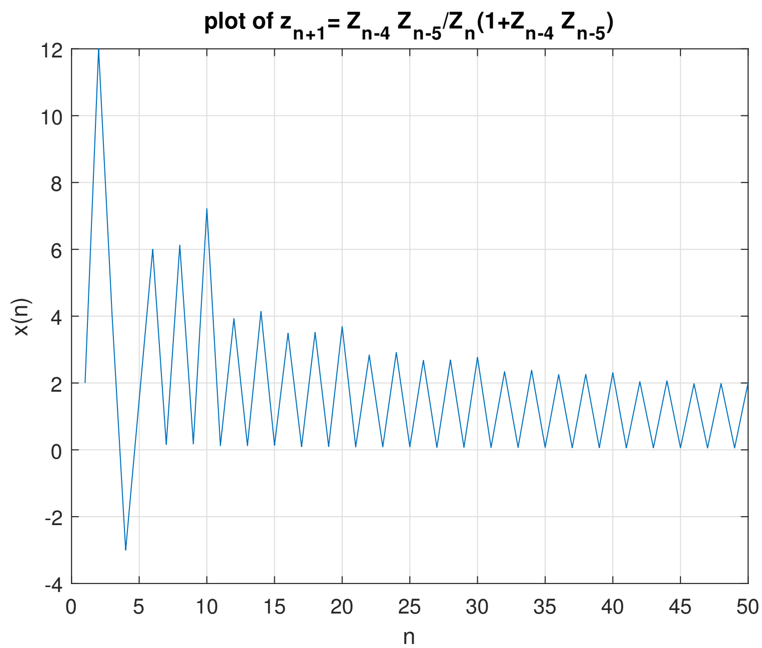

Figure 1:

we see that,

$$F_x(z^*, z^*, z^*) = -1,\ F_y(z^*, z^*, z^*) = 1,\ F_w(z^*, z^*, z^*) = 1.$$

By using Theorem 1 , the proof is completed. □

Numerical Example:

Now, To illustrate different types of solution of Eq.(3.1), we present numerical example.

Example 1. *Put* $z_{-5} = 2$, $z_{-4} = 12$, $z_{-3} = 4$, $z_{-2} = -3$, $z_{-1} = 1.5$, $z_0 = 6$ *in Eq.(3.1) see Figure 1.*

4 Qualitative Behavior of Solutions of $z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(-1+z_{n-4}z_{n-5})}$

In this part, we introduce the solutions of the following difference equation

$$z_{n+1}=\frac{z_{n-4}z_{n-5}}{z_n(-1+z_{n-4}z_{n-5})}, \quad n = 0, 1, 2, ..., \tag{4.1}$$

where the initial conditions $z_{-5},\ z_{-4},\ z_{-3},\ z_{-2,}\ z_{-1}$ and z_0 are arbitrary non-zero real numbers with $z_{-1}z_0, z_{-2}z_{-1}, z_{-3}z_{-2}, z_{-3}z_{-4}, z_{-5}z_{-4} \neq 1$.

Theorem 4. *Let $\{z_n\}_{n=-5}^{\infty}$ be a solution of difference equation (4.1). Then Eq.(4.1) has the following solutions for $n = 0, 1, ...$*

$$z_{10n-5} = \frac{f(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n}, \quad z_{10n-4} = \frac{e(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n},$$

$$z_{10n-3} = \frac{d(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n}, \quad z_{10n-2} = \frac{c(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n},$$

$$z_{10n-1} = \frac{b(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n}, \quad z_{10n} = \frac{a(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n},$$

$$z_{10n+1} = \frac{ef(de-1)^n(bc-1)^n}{a(ef-1)^{n+1}(cd-1)^n(ab-1)^n}, \quad z_{10n+2} = \frac{da(ef-1)^{n+1}(cd-1)^n(ab-1)^n}{f(de-1)^{n+1}(bc-1)^n},$$

$$z_{10n+3} = \frac{cf(de-1)^{n+1}(bc-1)^n}{a(ef-1)^{n+1}(cd-1)^{n+1}(ab-1)^n}, \quad z_{10n+4} = \frac{ba(ef-1)^{n+1}(cd-1)^{n+1}(ab-1)^n}{f(de-1)^{n+1}(bc-1)^{n+1}},$$

where $z_{-5} = f$, $z_{-4} = e$, $z_{-3} = d$, $z_{-2} = c$, $z_{-1} = b$, $z_0 = a$.

Proof. For $n = 0$, the result holds. Now, assume that $n > 0$ and that our assumption holds for $n - 1$. That is

$$z_{10n-15} = \frac{f(de-1)^{n-1}(bc-1)^{n-1}}{(ef-1)^{n-1}(cd-1)^{n-1}(ab-1)^{n-1}}, \quad z_{10n-14} = \frac{e(ef-1)^{n-1}(cd-1)^{n-1}(ab-1)^{n-1}}{(de-1)^{n-1}(bc-1)^{n-1}},$$

$$z_{10n-13} = \frac{d(de-1)^{n-1}(bc-1)^{n-1}}{(ef-1)^{n-1}(cd-1)^{n-1}(ab-1)^{n-1}}, \quad z_{10n-12} = \frac{c(ef-1)^{n-1}(cd-1)^{n-1}(ab-1)^{n-1}}{(de-1)^{n-1}(bc-1)^{n-1}},$$

$$z_{10n-11} = \frac{b(de-1)^{n-1}(bc-1)^{n-1}}{(ef-1)^{n-1}(cd-1)^{n-1}(ab-1)^{n-1}}, \quad z_{10n-10} = \frac{a(ef-1)^{n-1}(cd-1)^{n-1}(ab-1)^{n-1}}{(de-1)^{n-1}(bc-1)^{n-1}},$$

$$z_{10n-9} = \frac{ef(de-1)^{n-1}(bc-1)^{n-1}}{a(ef-1)^n(cd-1)^{n-1}(ab-1)^{n-1}}, \quad z_{10n-8} = \frac{da(ef-1)^n(cd-1)^{n-1}(ab-1)^{n-1}}{f(de-1)^n(bc-1)^{n-1}},$$

$$z_{10n-7}=\frac{cf(de-1)^n(bc-1)^{n-1}}{a(ef-1)^n(cd-1)^n(ab-1)^{n-1}},\quad z_{10n-6}=\frac{ba(ef-1)^n(cd-1)^n(ab-1)^{n-1}}{f(de-1)^n(bc-1)^n}.$$

From Eq.(4.1) that

$$\begin{aligned} z_{10n-5} &= \frac{z_{10n-10}z_{10n-11}}{z_{10n-6}(-1+z_{10n-10}z_{10n-11})} \\ &= \frac{ab}{\frac{ba(ef-1)^n(cd-1)^n(ab-1)^{n-1}}{f(de-1)^n(bc-1)^n}(-1+ab)} \\ &= \frac{f(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n}. \end{aligned}$$

$$\begin{aligned} z_{10n-4} &= \frac{z_{10n-9}z_{10n-10}}{z_{10n-5}(-1+z_{10n-9}z_{10n-10})} \\ &= \frac{\frac{ef}{(-1+ef)}}{\frac{f(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n}(-1+\frac{ef}{(-1+ef)})} \\ &= \frac{e(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n}. \end{aligned}$$

Similarly, we can proved other relations. □

Theorem 5. *The difference equation (4.1) has a periodic solution of periodic ten iff $cd=2$, $a=c$ and $d=f$ and we wilt get the form*

$$\{f,e,d,c,b,a,\frac{ef}{a(ef-1)},\frac{da}{f},\frac{cf}{a},\frac{ba}{f(ba-1)},f,e,d,c,...\}.$$

Proof. Assume that there exists a prime period ten solution of Eq.(4.1)

$$\{f,e,d,c,b,a,\frac{ef}{a(ef-1)},\frac{da}{f},\frac{cf}{a},\frac{ba}{f(ba-1)},...\},$$

from the solutions form of Eq.(4.1), we get

$$z_{10n-5}=\frac{f(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n}=f,$$

$$z_{10n-4}=\frac{e(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n}=e,$$

$$z_{10n-3} = \frac{d(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n} = d,$$

$$z_{10n-2} = \frac{c(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n} = c,$$

$$z_{10n-1} = \frac{b(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n} = b,$$

$$z_{10n} = \frac{a(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n} = a,$$

$$z_{10n+1} = \frac{ef(de-1)^n(bc-1)^n}{a(ef-1)^{n+1}(cd-1)^n(ab-1)^n} = \frac{ef}{a(ef-1)},$$

$$z_{10n+2} = \frac{da(ef-1)^{n+1}(cd-1)^n(ab-1)^n}{f(de-1)^{n+1}(bc-1)^n} = \frac{da}{f},$$

$$z_{10n+3} = \frac{cf(de-1)^{n+1}(bc-1)^n}{a(ef-1)^{n+1}(cd-1)^{n+1}(ab-1)^n} = \frac{cf}{a},$$

$$z_{10n+4} = \frac{ba(ef-1)^{n+1}(cd-1)^{n+1}(ab-1)^n}{f(de-1)^{n+1}(bc-1)^{n+1}} = \frac{ba}{f(ba-1)},$$

then

$$cd = 2, \quad a = c \ \ and \ \ d = f.$$

Second assume that $cd = 2, \quad a = c \ \ and \ \ d = f$. Then we see from the form of the solution of Eq.(4.1) that

$$z_{10n-5} = \frac{f(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n} = \frac{f(ef-1)^n(bc-1)^n}{(ef-1)^n(2-1)^n(bc-1)^n} = f,$$

$$z_{10n-4} = \frac{e(ef-1)^n(cd-1)^n(ab-1)^n}{(de-1)^n(bc-1)^n} = \frac{e(ef-1)^n(2-1)^n(ab-1)^n}{(ef-1)^n(ab-1)^n} = e,$$

$$z_{10n-3} = \frac{d(de-1)^n(bc-1)^n}{(ef-1)^n(cd-1)^n(ab-1)^n} = \frac{d(ef-1)^n(ab-1)^n}{(ef-1)^n(2-1)^n(ab-1)^n} = d.$$

Similarly, we can do the other relations. □

Theorem 6. *The equilibrium points of Eq.(4.1) are* $0, \pm\sqrt{2}$ *and they are not locally asymptotically stable.*

Proof. The equilibrium point of Eq.(4.1) is given by

$$z^* = \frac{{z^*}^2}{z^*(-1 + {z^*}^2)},$$

then

$$(2 - {z^*}^2){z^*}^2 = 0,$$

then $0, \pm\sqrt{2}$ are the equilibrium points

Define the function F by

$$F(x, y, w) = \frac{yw}{x(-1 + yw)}.$$

Then it follows that

$$\begin{aligned} F_x(x, y, w) &= \frac{-yw}{x^2(-1 + yw)}, \; F_y(x, y, w) = \frac{-w}{x(-1 + yw)^2}, \\ F_w(x, y, w) &= \frac{-y}{x(-1 + yw)^2}. \end{aligned}$$

We see that

$$F_x(z^*, z^*, z^*) = -1, \; F_y(z^*, z^*, z^*) = -1, \; F_w(z^*, z^*, z^*) = -1.$$

By using Theorem (1), the proof is completed. □

Numerical Example:

We present some numerical examples that illustrate different types of solutions of Eq.(4.1).

Example 2. *See Figure 2, when we take* $z_{-5} = 5$, $z_{-4} = 14$, $z_{-3} = 6$, $z_{-2} = 0.2$, $z_{-1} = 1$, $z_0 = 7$.

Example 3. *See Figure 3, if we put* $z_{-5} = 10$, $z_{-4} = 14$, $z_{-3} = 10$, $z_{-2} = 0.2$, $z_{-1} = -8$, $z_0 = 0.2$.

The following cases can be treated similarly:-

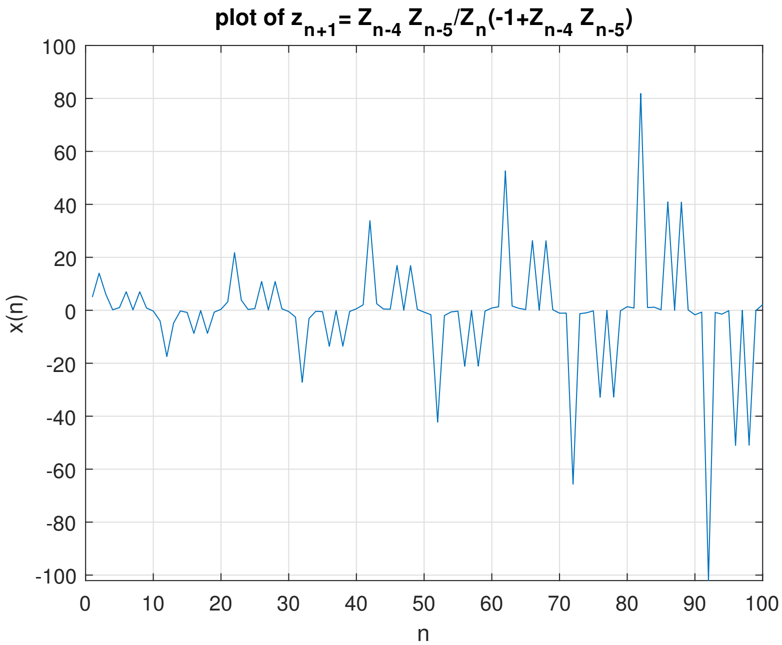

Figure 2:

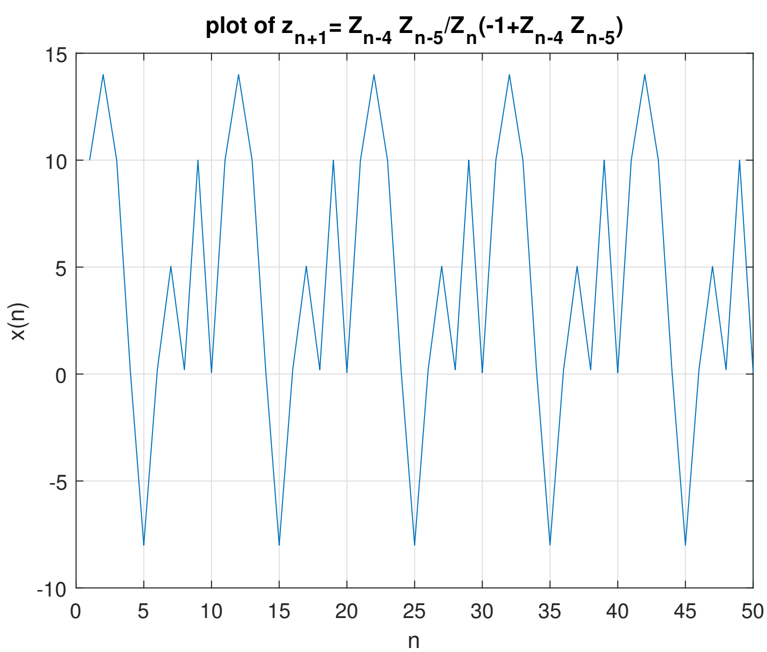

Figure 3:

5 Qualitative Behavior of Solutions of $z_{n+1}=\frac{z_{n-4}z_{n-5}}{z_n(1-z_{n-4}z_{n-5})}$

In this section, we obtain the expressions for the solution of the difference equation of the form:

$$z_{n+1}=\frac{z_{n-4}z_{n-5}}{z_n(1-z_{n-4}z_{n-5})}, \quad n=0,1,2,..., \tag{5.1}$$

where the initial conditions $z_{-5},\ z_{-4},\ z_{-3},\ z_{-2},\ z_{-1}$ and z_0 are arbitrary non-zero real numbers.

Theorem 7. *Let $\{z_n\}_{n=-5}^{\infty}$ be a solution of difference equation (5.1). Then for $n=0,1,...,$*

$$z_{10n-5}=f\prod_{j=0}^{n-1}\frac{[1-2jab][1-2jcd][1-2jef][1-(2j+1)bc][1-(2j+1)de]}{[1-(2j+1)ab][(1-(2j+1)cd][1-(2j+1)ef][1-2jbc][1-2jde]},$$

$$z_{10n-4}=e\prod_{j=0}^{n-1}\frac{[1-(2j+1)ab][(1-(2j+1)cd][1-(2j+1)ef][1-2jbc][1-2jde]}{[1-2jab][1-2jcd][1-2(j+1)ef][1-(2j+1)bc][1-(2j+1)de]},$$

$$z_{10n-3}=d\prod_{j=0}^{n-1}\frac{[1-2jab][1-2jcd][1-2(j+1)ef][1-(2j+1)bc][1-(2j+1)de]}{[1-(2j+1)ab][(1-(2j+1)cd][1-(2j+1)ef][1-2jbc][1-2(j+1)de]},$$

$$z_{10n-2}=c\prod_{j=0}^{n-1}\frac{[1-(2j+1)ab][(1-(2j+1)cd][1-(2j+1)ef][1-2jbc][1-2jde]}{[1-2jab][1-2(j+1)cd][1-2(j+1)ef][1-(2j+1)bc][1-(2j+1)de]},$$

$$z_{10n-1}=b\prod_{j=0}^{n-1}\frac{[1-2jab][1-2(j+1)cd][1-2(j+1)ef][1-(2j+1)bc][1-(2j+1)de]}{[1-(2j+1)ab][(1-(2j+1)cd][1-(2j+1)ef][1-2(j+1)bc][1-2(j+1)de]},$$

$$z_{10n}=a\prod_{j=0}^{n-1}\frac{[1-(2j+1)ab][(1-(2j+1)cd][1-(2j+1)ef][1-2(j+1)bc][1-2(j+1)de]}{[1-2(j+1)ab][1-2(j+1)cd][1-2(j+1)ef][1-(2j+1)bc][1-(2j+1)de]},$$

$$\begin{aligned} z_{10n+1} &= \frac{ef}{a(ef+1)}\prod_{j=0}^{n-1}\frac{[1-2(j+1)ab][1-2(j+1)cd][1-2(j+1)ef][1-(2j+1)bc]}{[1-(2j+1)ab][1-(2j+1)cd][1-(2j+3)ef][1-2(j+1)bc]} \\ &\times\frac{[1-(2j+1)de]}{[1-2(j+1)de]}, \end{aligned}$$

$$z_{10n+2} = \frac{da(1-ef)}{f(1-de)}\prod_{j=0}^{n-1}\frac{[1-(2j+1)ab][1-(2j+1)cd][1-(2j+3)ef][1-2(j+1)bc]}{[1-2(j+1)ab][1-2(j+1)cd][1-2(j+1)ef][1-(2j+1)bc]} \times\frac{[1-2(j+1)de]}{[1-(2j+3)de]},$$

$$z_{10n+3} = \frac{cf(1-de)}{a(1-ef)(1-cd)}\prod_{j=0}^{n-1}\frac{[1-2(j+1)ab][1-2(j+1)cd][1-2(j+1)ef]}{[1-(2j+1)ab][1-(2j+3)cd][1-(2j+3)ef]} \times\frac{[1-(2j+1)bc][1-(2j+3)de]}{[1-2(j+1)bc][1-2(j+1)de]},$$

$$z_{10n+4} = \frac{ba(1-ef)(1-cd)}{f(1-de)(1-bc)}\prod_{j=0}^{n-1}\frac{[1-(2j+1)ab][1-(2j+3)cd][1-(2j+3)ef]}{[1-2(j+1)ab][1-2(j+1)cd][1-2(j+1)ef]} \times\frac{[1-2(j+1)bc][1-2(j+1)de]}{[1-(2j+3)bc+][1-(2j+3)de]}.$$

where $x_{-5} = f, x_{-4} = e, x_{-3} = d, x_{-2} = c, x_{-1} = b, x_0 = a$.

Theorem 8. *The equilibrium point of Eq.(5.1) is* 0 *and it is not locally asymptotically stable.*

Example 4. *If we put the initial conditions of Eq.(5.1) as follows:* $z_{-5} = 3$, $z_{-4} = 13$, $z_{-3} = 9$, $z_{-2} = -4$, $z_{-1} = 2$, $z_0 = 8$. *See the following Figure:*

6 Qualitative Behavior of Solutions of $z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(-1-z_{n-4}z_{n-5})}$

In this section, we obtain the solution of the following difference equation

$$z_{n+1}=\frac{z_{n-4}z_{n-5}}{z_n(-1-z_{n-4}z_{n-5})}, \tag{6.1}$$

where the initial conditions $z_{-5},\ z_{-4},\ z_{-3},\ z_{-2},\ z_{-1}$ and z_0 are arbitrary non-zero real numbers.

Theorem 9. *Suppose that* $\{z_n\}$ *be a solution of Eq.(6.1) where the initial value* $z_{-5},\ z_{-4},\ z_{-3},\ z_{-2},\ z_{-1}$ *and* z_0 *are non-zero real numbers with* $z_{-1}z_0,\ z_{-2}z_{-1},\ z_{-3}z_{-2},\ z_{-3}z_{-4},\ z_{-5}z_{-4} \neq -1$. *Then the solution of Eq.(6.1) have the form*

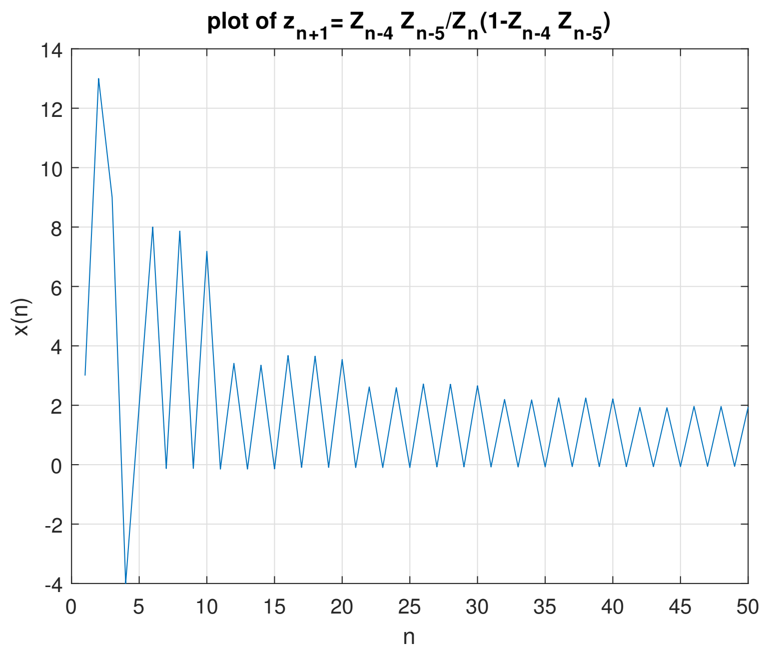

Figure 4:

$$z_{10n-5} = \frac{f(-de-1)^n(-bc-1)^n}{(-ef-1)^n(-cd-1)^n(-ab-1)^n},$$

$$z_{10n-4} = \frac{e(-ef-1)^n(-cd-1)^n(-ab-1)^n}{(-de-1)^n(-bc-1)^n},$$

$$z_{10n-3} = \frac{d(-de-1)^n(-bc-1)^n}{(-ef-1)^n(-cd-1)^n(-ab-1)^n},$$

$$z_{10n-2} = \frac{c(-ef-1)^n(-cd-1)^n(-ab-1)^n}{(-de-1)^n(-bc-1)^n},$$

$$z_{10n-1} = \frac{b(-de-1)^n(-bc-1)^n}{(-ef-1)^n(-cd-1)^n(-ab-1)^n},$$

$$z_{10n} = \frac{a(-ef-1)^n(-cd-1)^n(-ab-1)^n}{(-de-1)^n(-bc-1)^n},$$

$$z_{10n+1} = \frac{ef(-de-1)^n(-bc-1)^n}{a(-ef-1)^{n+1}(-cd-1)^n(-ab-1)^n},$$

$$z_{10n+2} = \frac{da(-ef-1)^{n+1}(-cd-1)^{n}(-ab-1)^{n}}{f(-de-1)^{n+1}(-bc-1)^{n}},$$

$$z_{10n+3} = \frac{cf(-de-1)^{n+1}(-bc-1)^{n}}{a(-ef-1)^{n+1}(-cd-1)^{n+1}(-ab-1)^{n}},$$

$$z_{10n+4} = \frac{ba(-ef-1)^{n+1}(-cd-1)^{n+1}(-ab-1)^{n}}{f(-de-1)^{n+1}(-bc-1)^{n+1}},$$

where $z_{-5} = f$, $z_{-4} = e$, $z_{-3} = d$, $z_{-2} = c$, $z_{-1} = b$, $z_0 = a$.

Theorem 10. *The difference equation (6.1) has equilibrium point which is* 0 *and it is not locally asymptotically stable.*

Theorem 11. *The difference equation (6.1) has a periodic solution of periodic ten iff* $cd = -2$, $a = c$ *and* $d = f$ *and takes the form*

$$\{f, e, d, c, b, a, \frac{ef}{a(-ef-1)}, \frac{da}{f}, \frac{cf}{a}, \frac{ba}{f(-ba-1)}, f, e, d, c, b, ...\}.$$

Example 5. *The following Figure shows the behavior of the solutions of Eq.(6.1) since* $z_{-5} = 8$, $z_{-4} = 15$, $z_{-3} = 4$, $z_{-2} = -2$, $z_{-1} = 5$, $z_0 = 6$.

Example 6. *Figure 6 shows the period ten solutions of Eq.(6.1) since* $z_{-5} = -8$, $z_{-4} = 15$, $z_{-3} = -8$, $z_{-2} = 0.25$, $z_{-1} = 5$, $z_0 = 0.25$.

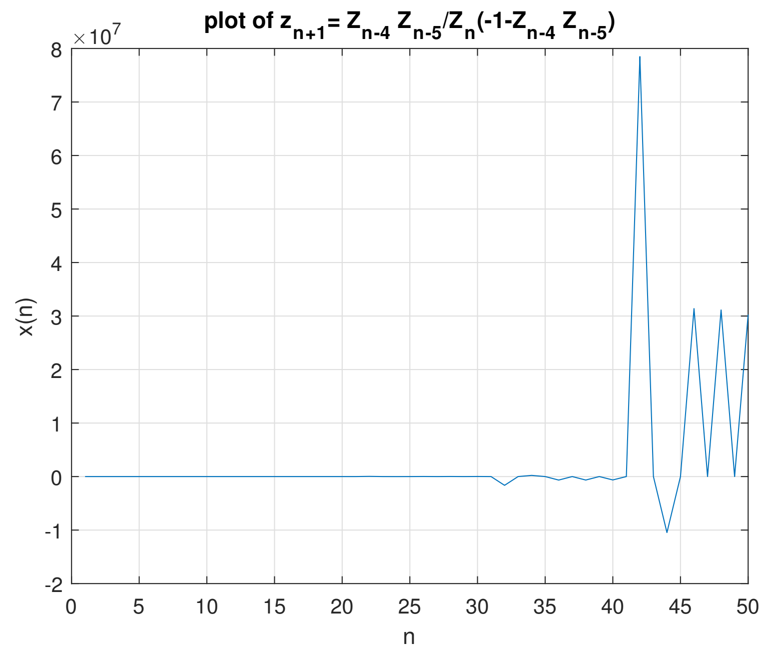

Figure 5:

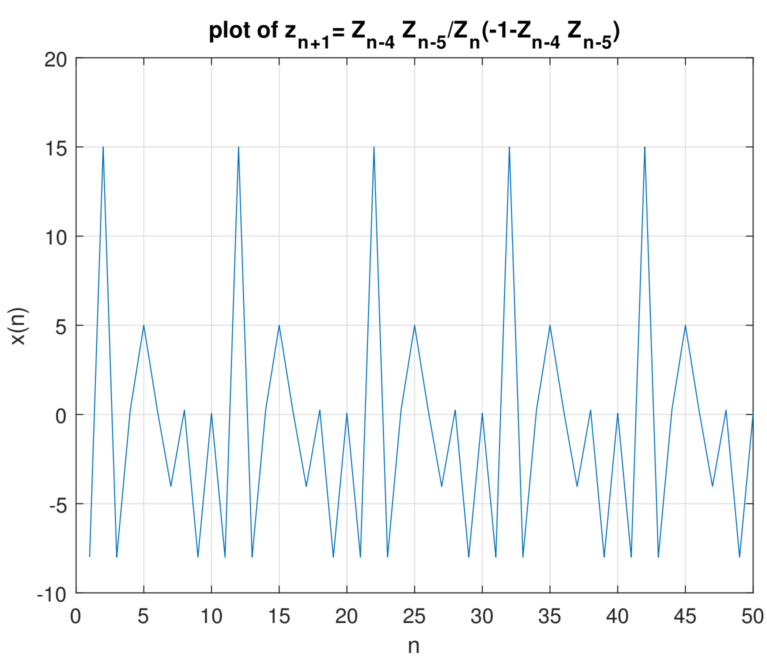

Figure 6:

Conclusion 1. *This paper discussed the expression's form of some rational six order difference equations. In section 3, we study some properties of the following recursive equation in the form* $z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(1+z_{n-4}z_{n-5})}$, *, first we have got the form of the solutions of this equation, studied the equilibrium point and gave numerical example and drew it by using Matlab. In Section 4, we have got the solution's of the equation* $z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(-1+z_{n-4}z_{n-5})}$*and take numerical examples. In Sections 5–6, we obtained the solution of the following equationes respectively,* $z_{n+1}=\frac{z_{n-4}z_{n-5}}{z_n(1-z_{n-4}z_{n-5})}$, $z_{n+1} = \frac{z_{n-4}z_{n-5}}{z_n(-1-z_{n-4}z_{n-5})}$. *Also, in each case we take numerical examples to illustrates the results.*

References

[1] H. S. Alayachi, M. S. M. Noorani and E. M. Elsayed, Qualitative analysis of a fourth order difference equation, Journal of Applied Analysis and Computation, 10 (4) (2020), 1343-1354.

[2] M. Aloqeili, Dynamics of a Rational Difference Equation, Applied Mathematics and Computation, 176 (2) (2006), 768-778.

[3] M. Almatrafi, E. Elsayed and F. Alzahrani, Qualitative behavior of a quadratic second-order rational difference equation, International Journal of Advanced in Mathematics, 2019 (1) (2019), 1-14.

[4] M. B. Almatrafi, E. M. Elsayed and F. Alzahrani, The Solution and Dynamic Behavior of Some Difference Equations of Fourth Order, DCDIS: Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 29 (1a) (2022), 33-50.

[5] A. M. Alotaibi, M. S. M. Noorani and M. A. El-Moneam, On the Solutions of a System of Third-Order Rational Difference Equations, Discrete Dynamics in Nature and Society, 2018 (2018), Article ID 1743540.

[6] I. M. Alsulami and E. M. Elsayed, On the dynamical behaviors and periodicity of difference equation of order three, Journal of New Results in Science, 11 (1) (2022), 48-61.

[7] M. Alzubaidi and E. M. Elsayed, Analytical and Solutions of Fourth Order Difference Equations, Communications in Advanced Mathematical Sciences, 2 (1) (2019), 9-21.

[8] N. Battaloglu, C. Cinar and I. Yalçınkaya, The dynamics of the difference equation, ARS Combinatoria, 97 (2010), 281-288.

[9] S. E. Das and M. Bayram, On a system of rational difference equations, World Applied Sciences Journal, 10 (11) (2010), 1306–1312.

[10] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Qualitative behavior of higher order difference equation, Soochow Journal of Mathematics, 33 (4) (2007), 861-873.

[11] M. M. El-Dessoky, On a solvable for some nonlinear systems of difference equations, J. Comput. Theor. Nanosci., 12 (2015), 3432-3442.

[12] H. El-Metwally, Solutions form of some rational systems of difference equations. Discrete Dynamics in Nature and Society, 2017 (2017) Article ID 903593.

[13] M. El-Moneam, T. Ibrahim and S. Elamody, Stability of a fractional difference equation of higher-order, Journal of Nonlinear Sciences and Applications, 12 (2019), 65-74.

[14] E. M. Elsayed, Dynamics and Behavior of a Higher Order Rational Difference Equation, Journal of Nonlinear Sciences and Applications, 9 (4) (2016), 1463-1474.

[15] E. Elsayed and M. El-Dessoky, Dynamics and global behavior for a forth order rational difference equation, Hacettepe Journal of Mathematics and Statistics, 42 (5) (2013), 479 – 494.

[16] E. Elsayed, M. El-Dessoky and A. Asiri, Dynamics and behavior of a second order rational difference equation, Journal of Computational Analysis and Applications, 16 (4) (2014), 794-807.

[17] E. M. Elsayed and F. Alzahrani, Periodicity and solutions of some rational difference equations systems, Journal of Applied Analysis and Computation 9 (6) (2019), 2358-2380.

[18] E. M. Elsayed, A. Alshareef and F. Alzahrani, Qualitative behavior and solution of a system of three-dimensional rational difference equations, Mathematical Methods in the Applied Sciences, 45 (9) (2022), 5456–5470.

[19] E. M. Elsayed, B. S. Aloufi and O. Moaaz, The Behavior and Structures of Solution of Fifth-Order Rational Recursive Sequence, Symmetry, 2022, 14, 641, 1-18.

[20] E. M. Elsayed, B. S. Alofi and A. Q. Khan, Qualitative Behavior of Solutions of Tenth-Order Recursive Sequence Equation, Mathematical Problems in Engineering, Volume 2022 (2022), Article ID 5242325, 10 pages.

[21] E. M. Elsayed and N. H. Alotaibi, The Pattern and Conduct of the Solutions of Certain Nonlinear Difference Equations, Journal of Pure and Applied Mathematics: Advances and Applications, 24 (1) (2021), 21-57.

[22] E. M. Elsayed and K. N. Alharbi, The expressions and behavior of solutions for nonlinear systems of rational difference equations, Journal of Innovative Applied Mathematics and Computational Sciences (JIAMCS), 2 (1) (2022), 78–91.

[23] E. M. Elsayed and A. Alshareef, Qualitative Behavior of A System of Second Order Difference Equations, European Journal of Mathematics and Applications, 1:15 (2021), 1-11.

[24] E. M. Elsayed and H. S. Gafel, Some Systems of Three Nonlinear Difference Equations, Journal of Computational Analysis and Applications, 29 (1) (2021), 86-108.

[25] E. M. Elsayed and N. H. Alotaibi, The Form of the Solutions and Behavior of Some Systems of Nonlinear Difference Equations, Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis, 27 (2020), 283-297.

[26] E. M. Elsayed, J. G. Al-Juaid and H. Malaikah, On the dynamical behaviors of a quadratic difference equation of order three, European Journal of Mathematics and Applications, 3 (2023), Article ID 1.

[27] E. M. Elsayed and M. M. Alzubaidi, On a Higher-Order Systems of Difference Equations, Pure and Applicable Analysis, 2023 (2023): 2.

[28] E. M. Elsayed and J. G. AL-Juaid, The Form of Solutions and Periodic Nature for Some System of Difference Equations, Fundamental Journal of Mathematics and Applications, 6 (1) (2023), 24-34.

[29] E. M. Elsayed and B. Alofi, Stability analysis and periodictly properties of a class of rational difference equations, MANAS Journal of Engineering, 10 (2) (2022), 203-210.

[30] E. M. Elasyed and M. T. Alharthi, The form of the solutions of fourth order rational systems of difference equations, Annals of Communications in Mathematics, 5 (3) (2022), 161-180.

[31] E. M. Elsayed, and H. S. Gafel, Dynamics and Global Stability of Second Order Nonlinear Difference Equation, Pan-American Journal of Mathematics, 1 (2022), 16.

[32] E. M. Elsayed, J. G. Al-Juaid and H. Malaikah, On the dynamical behaviors of a quadratic difference equation of order three, European Journal of Mathematics and Applications, 3 (2023), Article ID 1.

[33] E. M. Elsayed and F. A. Al-Rakhami, On Dynamics and Solutions Expressions of Higher-Order Rational Difference Equations, Ikonion Journal of Mathematics, 5(1) (2023), 39-61.

[34] E. M. Elasyed and M. T. Alharthi, On the solutions and the periodicity of some rational difference equations systems, European Journal of Mathematics and Applications 3 (2023), Article ID 4.

[35] N. Haddad and N. Touafek, Solution form of a higher-order system of difference equations and dynamical behavior of its special case, Math. Methods Appl. Sci., 40 (2017), 3599–3607.

[36] N. Haddad and J. F. Rabago, Well-defined solutions of a system of difference equations, J. Appl. Math. Comput., 56 (2018), 439–458.

[37] T. F. Ibrahim and N. Touafek, Max-type system of difference equations with positive two-periodic sequences, Math. Methods Appl. Sci., 37 (16) (2014), 2562-2569.

[38] T. F. Ibrahim and A. Q. Khan, Forms of solutions for some two-dimensional systems of rational partial recursion equations, Mathematical Problems in Engineering, Volume 2021 (2021), Article ID 9966197, 10 pages.

[39] M. Gumus and R. Abo-Zeid, An explicit formula and forbidden set for a higher order difference equation, Journal of Applied Mathematics and Computing, 63 (2020), 133—142.

[40] R. Karataş, A. Gelişken, A solution form of a higher order difference equation, Korunalp Journal of Mathematics, 9(2) (2021), 316-323.

[41] R. Karataş, A. Gelişken, On a solvable difference equation with sequence coefficients, Advances and Applications in Discrete Mathematics, 30 (2022), 27-33.

[42] R. Karatas, Global behavior of a higher order difference equation, Computers Mathematics with Applications, 60 (2010), 830-839.

[43] R. Khalaf-Allah, Asymptotic Behaviour and Periodic Naturel of Two Difference Equations, Ukrainian Mathematical Journal, 61 (6) (2009), 988-993.

[44] A. Khaliq, F. Alzahrani and E. M. Elsayed, Global Attractivity of a Rational Difference Equation of Order Ten, Journal of Nonlinear Sciences and Applications, 9 (2016), 4465–4477.

[45] A. Khaliq and E. M. Elsayed, The dynamics and solution of some difference equations, J. Nonlinear Sci. Appl., 9(3) (2016), 1052-1063.

[46] A. Khaliq and M. Shoaib, Dynamics of three-dimentional system of second order rational difference equations, Electronic Journal of Mathematical Analysis and Applications, 9 (2) (2021), 308-319.

[47] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[48] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.

[49] A. S. Kurbanli, C. Cinar and I. Yalçınkaya, On the behavior of positive solutions of the system of rational difference equations, Mathematical and Computer Modelling, 53 (2011), 1261-1267.

[50] A. S. Kurbanli, On the behavior of solutions of the system of rational difference equations, Advances in Difference Equations 2011 (2011), 2011:40 doi:10.1186/1687-1847-2011-40.

[51] I. Okumuş and Y. Soykan, On the solutions of four second-order nonlinear difference equations, Universal Journal of Mathematics and Applications, 2 (3) (2019), 116-125.

[52] M. Ozair, U. Saeed, M. Ozair and T. Hussain, Fractional-order vector-host disease model, DCDIS, Series B: Applications & Algorithms, 24 (2017), 97-111.

[53] A. Sanbo and E. M. Elsayed, Analytical study of a system of difference equation, Asian Res. J. Math., 14 (1) (2019), 1-18.

[54] T. Tollou Y. Yazlik and N. Taskara, Behavior of positive solutions of a difference equation, J. Appl. Math. & Informatics, 35 (3-4) (2017), 217 - 230.

[55] H. Yacine, Form and periodicity of solutions of some systems of higher-order difference equations, Math. Sci. Lett., 5 (2016), 79-84.

[56] H. Yacine, N. Touafek and Y. Yazlik, Dynamic behavior of a second-order nonlinear rational difference equation, Turkish Journal of Mathematics, 39 (6) (2015), 1004-1018.

[57] I. Yalçınkaya, C. Cinar and D. Simsek, Global asymptotic stability of a system of difference equations, Applicable Analysis, 87 (6) (2008), 689-699.

[58] I. Yalcinkaya and C. Cinar, Global asymptotic stability of two nonlinear difference equations, Fasciculi Mathematici, 43 (2010), 171-180.

[59] Y. Yazlik, D. T. Tollu and N. Taskara, On the behaviour of solutions for some systems of difference equations, Journal of Computational Analysis and Applications, 18 (1) (2015), 166-178.

[60] Y. Yazlik, On the behaviour of the solutions of difference equation systems, J. Comput. Anal. Appl., 16 (5) (2014), 932–941.

[61] Q. Zhang, J. Liu and Z. Luo, Dynamical behavior of a system of third-order rational difference equation, Discrete Dyn. Nat. Soc., 2015 (2015), 9 pages.