On a Strongly Nonlinear Degenerate Elliptic Equations in Weighted Sobolev Spaces

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Abstract

In this paper, we prove the existence and uniqueness of weak solution to a strongly nonlinear degenerate elliptic problem of the type:

$$-\text{div}\Big[\omega_1 a(x, \nabla u) + \omega_2 b(x, u, \nabla u)\Big] + \omega_3 g(x) u(x) = f(x).$$

Here, ω_1, ω_2 and ω_3 are A_p-weight functions that will be defined in the preliminaries, where, Ω is a bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ and $f \in L^1(\Omega)$, with $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $a : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : \Omega \longrightarrow \mathbb{R}$ are functions that satisfy some conditions and f belongs to $L^{p'}(\Omega, \omega_1^{1-p'})$. First, we transformed the problem into an equivalent operator equation; second, we utilized the Browder-Minty Theorem to prove the existence and uniqueness of weak solution to the considered problem.

Keywords: Strongly nonlinear degenerate elliptic equations, Browder-Minty theorem, Dirichlet problem, Weighted Sobolev spaces, Weak solution.

1 Introduction

Let Ω be a bounded open subset in $\mathbb{R}^n$ ($n \geq 2$), $\partial\Omega$ its boundary and $p > 1$ and ω_1, ω_2 and ω_3 are a weights functions in Ω (ω_1, ω_2 and ω_3 are measurable and strictly positive a.e. in Ω).

In this paper, we consider the following problem:

$$\begin{cases} \mathcal{L}u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{L}$ is given by

$$\mathcal{L}u(x) = -\text{div}\Big[\omega_1(x)a\Big(x, \nabla u(x)\Big) \;+\; \omega_2(x)b\Big(x, u(x), \nabla u(x)\Big)\Big] \;+\; \omega_3(x)g(x)u(x), \tag{2}$$

with $f \in L^1(\Omega)$. Furthermore, the operators $a : \Omega\times\mathbb{R}^n \longrightarrow \mathbb{R}$ and $b : \Omega\times\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ are Carathéodory function satisfying the assumptions of growth, ellipticity and monotonicity, and the nonlinear term $g : \Omega \longrightarrow \mathbb{R}$ is a positive function.

In the past decade, much attention has been devoted to nonlinear elliptic equations because of their wide application to physical models such as non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous model, celestial mechanics and reaction-diffusion problems (we refer to [4,9,27] where it is possible to find some examples of applications of degenerate elliptic equations). One of the motivations for studying (1) comes from applications to electrorheological fluids (see [24] for more details) as an important class of non-Newtonian fluids.

Many scholars have examined equations like (1), where $a(x, \nabla u) \equiv g(x) \equiv 0$ and $\omega_1 \equiv \omega_2 \equiv \omega_3 \equiv 1$ (see [8, 22] and the references therein). The degenerate case with difierent conditions haven been studied by many authors (we refer to [1, 5, 6, 20–23] for more details).

Recently, Drábek and al. [10] proved that under some additional assumptions on a and h, the problem $-div(a(x, u, \nabla u)) = h$ has a solution $u \in W_0^{1,p}(\Omega, \omega)$. Moreover in [7], the author proved the existence of solution for Problem (1), when the nonlinear term $g(x) \equiv 0$.

Our goal in this research is to study (1) in $W_0^{1,p}(\Omega, \omega_1)$. We will use the Browder-Minty Theorem and the weighted Sobolev spaces theory to prove that (1) has a unique weak solution $u \in W_0^{1,p}(\Omega, \omega_1)$.

In terms of our Problem's existence, there are many difficulties associated with this kind of problems. Firstly, the operator $\mathcal{L}$ can not be defined from $W_0^{1,p}(\Omega, \omega_1)$ into its dual space $[W_0^{1,p}(\Omega, \omega_1)]^*$. The second difficulty is establish the relationship between ω_1, ω_2 and ω_3, in order to ensure the existence and uniqueness of solution for Problem (1).

Let us speedily summarize the work's contents. In Section 2, we provide some preliminary information as well as certain lemmas. In Section 3, we specify all of the assumptions on a, b, g

and we introduce the notion of weak solution for the Problem (1). The main results will be stated and proved in Section 4. Section 5 contains an example that exemplifies our principal result.

2 Preliminaries

In this section,we give some preliminaries facts which are used throughout this paper. Monographs by J. Garcia-Cuerva and J. L. Rubio de Prancia [14] and A. Torchinsky [25] have comprehensive expositions.

Let v be a weight function in $\mathbb{R}^N$, that is ω measurable and strictly positive a.e. in $\mathbb{R}^N$. For $1 \le p < \infty$, we denote by $L^p(\Omega, v)$ the space of measurable functions u on Ω such that

$$||u||_{L^p(\Omega,v)} = \left(\int_\Omega |u(x)|^p v(x)dx\right)^{\frac{1}{p}} < \infty,$$

where Ω be open in $\mathbb{R}^n$. It is a well-known fact that the space $L^p(\Omega, \omega)$, endowed with this norm is a Banach space. We also have that the dual space of $L^p(\Omega, v)$ is the space $L^{p'}(\Omega, v^{1-p'})$.

Proposition 1. *[17, 18] Let* $1 \le p < \infty$. *If*

$$v^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \qquad if \quad p > 1,$$

$$ess \sup_{x\in B} \frac{1}{v(x)} < +\infty \qquad if \quad p = 1,$$

for every ball $B \subset \Omega$. *Then,*

$$L^p(\Omega, v) \subset L^1_{loc}(\Omega).$$

As a consequence, under conditions of Proposition 1, the convergence in $L^p(\Omega, v)$ implies convergence in $L^1_{loc}(\Omega)$. Moreover, every function in $L^p(\Omega, v)$ has a distributional derivatives.

Definition 1. *[18, 19] Let* $1 < p < \infty$. *A weight* v *is siad to be an* A_p*-weight if there exists* $A = A(p, \omega)$ *such that*

$$\left(\frac{1}{|B|}\int_B v(x)dx\right)\left(\frac{1}{|B|}\int_B \left(v(x)\right)^{\frac{-1}{p-1}} dx\right)^{p-1} \le A,$$

for all $B \subset \mathbb{R}^n$, *where* $|B|$ *denotes the n-dimensional Lebesgue measure of* B *in* $\mathbb{R}^n$.

The infimum over all such constants A is called the A_p constant of ω. We denote by A_p, $1 < p < \infty$, the set of all A_p weights.
If $1 \leq q \leq p < \infty$, then $A_1 \subset A_q \subset A_p$ (we refer to [15, 16, 26] for complete details about A_p-weights).

Example 1. *(Example of A_p-weights)*

(i) *If $C \leq \omega(y) \leq D$ for a.e. $y \in \mathbb{R}^n$, such that C and D two positive constants , then $\omega \in A_p$ for $1 < p < \infty$.*

(ii) *Suppose that $\omega(y) = |y|^\sigma$, $y \in \mathbb{R}^n$. Then $\omega \in A_p$ iff $-n < \sigma < n(p-1)$ for $1 < p < \infty$ (see Corollary 4.4 in [25]).*

(iii) *Let Ω be an open subset of $\mathbb{R}^n$. Then $\omega(y) = e^{\lambda v(y)} \in A_2$, with $v \in W^{1,n}(\Omega)$ and λ is sufficiently small (see Corollary 2.18 in [19]).*

Proposition 2. *[26] Let $v \in A_p$ with $1 \leqslant p < \infty$ and let F be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then*
$$\left(\frac{|F|}{|B|}\right)^p \leqslant C\frac{v(F)}{v(B)}$$
where C is the A_p constant of v.

The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined as follows.

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be open, and let ω be an A_p-weight, $1 < p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega, \omega)$ as the set of functions $u \in L^p(\Omega, \omega)$ with $D_k u \in L^p(\Omega, \omega)$, for $k = 1, ..., n$. The norm of u in $W^{1,p}(\Omega, \omega)$ is given by*
$$||u||_{W^{1,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x)dx + \sum_{k=1}^n \int_\Omega |D_k u(x)|^p \omega(x)dx\right)^{\frac{1}{p}}.$$
We also define $W_0^{1,p}(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \omega)$ with respect to the norm $||.||_{W^{1,p}(\Omega,\omega)}$. Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, \omega)$.

Equipped by this norm, $W^{1,p}(\Omega, \omega)$ and $W_0^{1,p}(\Omega, \omega)$ are separable and reflexive Banach spaces (see Proposition 2.1.2. in [17] and see [16, 18] for more informations about the spaces $W^{1,p}(\Omega, \omega)$). The dual of space $W_0^{1,p}(\Omega, \omega)$ is the space $W_0^{-1,p'}(\Omega, \omega^{1-p'})$, where
$$W_0^{-1,p'}(\Omega, \omega^{1-p'}) = \left\{f_0 - \text{div}(F) \ / \ F = (f_1, ..., f_n) \ : \ \frac{f_i}{\omega} \in L^{p'}(\Omega, \omega), \ i = 0, ..., n\right\}.$$

The following theorems will be needed throughout this paper(we refer to [11, 13, 28]).

Theorem 1. *Let $v \in A_p$, $1 < p < \infty$, and let $\Omega \subset \mathbb{R}^n$. If $u_k \longrightarrow u$ in $L^p(\Omega, v)$, then there exist a subsequence (u_{k_j}) and $\psi \in L^p(\Omega, v)$ such that*

(i) $u_{k_j}(y) \longrightarrow u(y)$, $k_j \longrightarrow \infty$, *$v$-a.e. on Ω.*

(ii) $|u_{k_j}(y)| \leq \psi(y)$, *$v$-a.e. on Ω.*

Theorem 2. *Let $v \in A_p$ and $\Omega \subset \mathbb{R}^n$. There exist θ, $\mu > 0$ such that for any $f \in W_0^{1,p}(\Omega, v)$ and each ν verifing $1 \leq \nu \leq \frac{n}{n-1} + \mu$,*

$$||f||_{L^{\nu p}(\Omega,v)} \leq \theta ||\nabla f||_{L^p(\Omega,v)},$$

where θ depends only on n, p, the A_p constant of v and the diameter of Ω.

The Browder-Minty Theorem is stated as follows.

Theorem 3. *Let $\mathcal{A} : W \longrightarrow W^*$ be a hemicontinuous, coercive, and monotone operator on reflexive and separable Banach space W. The following assertions are then true:*

(a) *$\mathcal{A}u = G$ has a solution $u \in W$ for all $G \in W^*$,*

(b) *If the operator $\mathcal{A}$ is strictly monotone, then the solution $u \in W$ is unique.*

3 Basic assumptions and notion of solutions

3.1 Basic assumptions

Let us now give the precise hypotheses on the Problem (1), and we make the following hypotheses: $\Omega \subset \mathbb{R}^n(\ n \geq 2)$, $1 < q, s < p < \infty$, and $\omega_i \in A_p$ for $i = 1, 2, 3$, and let $a : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and $g : \Omega \longrightarrow \mathbb{R}$ satisfying the following assumptions:

(A1) For $i = 1, ..., n$, b_i and a_i are Caratéodory functions, with

$$a(x, \zeta) = \Big(a_1(x, \zeta), ..., a_n(x, \zeta)\Big)$$

and

$$b(x, \sigma, \zeta) = \Big(b_1(x, \sigma, \zeta), ..., b_n(x, \sigma, \zeta)\Big).$$

(A2) There are positive functions $h_1,\ h_2,\ \tilde{h_2} \in L^{\infty}(\Omega)$ and $f_1 \in L^{p'}(\Omega, \omega_1)\Big(\text{with } \frac{1}{p} + \frac{1}{p'} = 1\Big)$ and $f_2 \in L^{q'}(\Omega, \omega_2)\Big(\text{with } \frac{1}{q} + \frac{1}{q'} = 1\Big)$ such that :

$$|a(x, \zeta)| \leq f_1(x) + h_1(x)|\zeta|^{p-1},$$

and

$$|b(x, \sigma, \zeta)| \leq f_2(x) + h_2(x)|\sigma|^{q-1} + \tilde{h_2}(x)|\zeta|^{q-1}.$$

(A3) There exists a constant $\alpha > 0$ such that :

$$\langle a(x, \zeta) - a(x, \zeta^{'}), \zeta - \zeta^{'} \rangle \geqslant \alpha|\zeta - \zeta^{'}|^{p},$$

and

$$\langle b(x, \sigma, \zeta) - b(x, \sigma^{'}, \zeta^{'}), \zeta - \zeta^{'} \rangle \geqslant 0,$$

whenever $(\sigma, \zeta), (\sigma', \zeta') \in \mathbb{R} \times \mathbb{R}^n$ with $\sigma \neq \sigma^{'}$ and $\zeta \neq \zeta^{'}$.

(A4) There are constants $\beta_1,\ \beta_2,\ \beta_3 > 0$ such that :

$$\langle a(x, \zeta), \zeta \rangle \geqslant \beta_1|\zeta|^{p},$$

and

$$\langle b(x, \sigma, \zeta), \zeta \rangle \geqslant \beta_2|\zeta|^{q} + \beta_3|\sigma|^{q}.$$

(A5) $g \in L^{p}(\Omega, \omega_3)$, with $\frac{1}{p} = \frac{1}{s'} - \frac{1}{s}$ and $g(x) \geqslant 0$.

3.2 Notions of solutions

Definition 3. *One says $u \in W_0^{1,p}(\Omega, \omega_1)$ is a weak solution to* (1)*, assuming that*

$$\int_{\Omega} a(x, \nabla u(x)).\nabla v(x)\omega_1(x)dx + \int_{\Omega} b(x, u(x), \nabla u(x)).\nabla v(x)\omega_2(x)dx + \int_{\Omega} g(x)u(x)v(x)\omega_3 dx$$
$$= \int_{\Omega} f(x)v(x)dx,$$

for every $v \in W_0^{1,p}(\Omega, \omega_1)$.

Remark 1. *We seek to establish a relationship between ω_1, ω_2 and ω_3, in order to ensure the existence and uniqueness of solution for our Problem* (1)*. At first we notice:*

(i) *If* $\frac{\omega_2}{\omega_1} \in L^{r_1}(\Omega, \omega_1)$ *where* $r_1 = \frac{p}{p-q}$, $1 < q < p < \infty$ *and* $\omega_1, \omega_2 \in A_p$*, then, by Hölder inequality we obtain*

$$||u||_{L^q(\Omega,\omega_2)} \leqslant C_{p,q}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,q} = ||\frac{\omega_2}{\omega_1}||^{1/q}_{L^{r_1}(\Omega,\omega_1)}$.

(ii) *Analogously, if* $\frac{\omega_3}{\omega_1} \in L^{r_2}(\Omega, \omega_1)$ *where* $r_2 = \frac{p}{p-s}$, $1 < s < p < \infty$ *and* $\omega_1, \omega_3 \in A_p$*, then*

$$||u||_{L^s(\Omega,\omega_3)} \leqslant C_{p,s}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,s} = ||\frac{\omega_3}{\omega_1}||^{1/s}_{L^{r_2}(\Omega,\omega_1)}$.

4 Main result

We are now in the position to get existence result of weak solutions for (1).

4.1 Result on the existence and uniqueness

Our main result is as follows.

Theorem 4. *Let* $\omega_i \in A_p (i = 1, 2, 3)$, $1 < q, s < p < \infty$, *and assume that the conditions* ***(A1)*** $-$ ***(A5)*** *holds. If* $\frac{f}{\omega_1} \in L^{p'}(\Omega, \omega_1)$, $\frac{\omega_2}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ *and* $\frac{\omega_3}{\omega_1} \in L^{p/(p-s)}(\Omega, \omega_1)$*, then* (1) *has a unique solution* $u \in W^{1,p}_0(\Omega, \omega_1)$. *Moreover, we have*

$$\|u\|_{W^{1,p}_0(\Omega,\omega_1)} \quad \leq \quad \varepsilon\left(||f/\omega_1||_{L^{p'}(\Omega,\omega_1)}\right)^{1/p-1},$$

where $\varepsilon = \left[\frac{\beta_1}{\theta^{p+1}+\theta}\right]^{1/1-p}$.

4.2 Proof of Theorem 4

The essential one of our proof is to reduce the (1) to an operator problem $\mathcal{A}u = \mathbf{G}$ and then using the Browder-Minty Theorem 3. The proof will be separated into five steps.

4.2.1 Equivalent operator equation

In this subsection, we use the some tools and the condition **(A2)** to prove an existence the operator $\mathcal{A}$ such that the Problem (1) is equivalent to the operator equation $\mathcal{A}u = \mathbf{G}$. We introduce the operators $\mathbf{G} : W_0^{1,p}(\Omega, \omega_1) \longrightarrow \mathbb{R}$ and $\Gamma : W_0^{1,p}(\Omega, \omega_1) \times W_0^{1,p}(\Omega, \omega_1) \longrightarrow \mathbb{R}$ such that

$$\mathbf{G}(v) = \int_\Omega f(x)v(x)dx,$$

and

$$\Gamma(u, v) = \Gamma_1(u, v) + \Gamma_2(u, v) + \Gamma_3(u, v),$$

where $\Gamma_i : W_0^{1,p}(\Omega, \omega_1) \times W_0^{1,p}(\Omega, \omega_1) \longrightarrow \mathbb{R}$, for $i = 1, 2, 3$, are defined as

$$\Gamma_1(u, v) = \int_\Omega a(x, \nabla u).\nabla v \omega_1 dx, \quad \Gamma_2(u, v) = \int_\Omega b(x, u, \nabla u).\nabla v \omega_2 dx,$$

$$\text{and} \quad \Gamma_3(u, v) = \int_\Omega g(x)u(x)v(x)\omega_3 dx.$$

Consequently, the weak formulation of (1) is given by the operator equation

$$\Gamma(u, v) = \mathbf{G}(v), \qquad \text{for all } v \in W_0^{1,p}(\Omega, \omega_1).$$

We will show that $\mathbf{G} \in W_0^{-1,p'}(\Omega, \omega_1^{1-p'})$ and $\Gamma(u, .)$ is linear and continuous, for each $u \in W_0^{1,p}(\Omega, \omega_1)$.

(i) Using Hölder inequality and Theorem 2, we obtain

$$\begin{aligned} |\mathbf{G}(v)| &\leq \int_\Omega \frac{|f|}{\omega_1} |v|\omega_1 \, dx \\ &\leq ||f/\omega_1||_{L^{p'}(\Omega,\omega_1)} ||v||_{L^p(\Omega,\omega_1)} \\ &\leq \theta ||f/\omega_1||_{L^{p'}(\Omega,\omega_1)} ||v||_{W_0^{1,p}(\Omega,\omega_1)}. \end{aligned}$$

Since $f/\omega_1 \in L^{p'}(\Omega, \omega_1)$, then $\mathbf{G} \in W_0^{-1,p'}(\Omega, \omega_1^{1-p'})$.

(ii) Let $u \in W_0^{1,p}(\Omega, \omega_1)$. We have

$$|\Gamma(u, v)| \leq |\Gamma_1(u, v)| + |\Gamma_2(u, v)| + |\Gamma_3(u, v)|. \tag{3}$$

On the other hand, we get by using **(A2)**, Hölder inequality, Remark 1 **(i)** and Theorem 2,

$$\begin{aligned}
|\Gamma_1(u,v)| &\leq \int_\Omega |a(x,\nabla u)||\nabla v|\omega_1 dx \\
&\leq \int_\Omega \left(f_1 + h_1|\nabla u|^{p-1}\right)|\nabla v|\omega_1 dx \\
&\leq ||f_1||_{L^{p'}(\Omega,\omega_1)}||\nabla v||_{L^p(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||\nabla u||^{p-1}_{L^p(\Omega,\omega_1)}||\nabla v||_{L^p(\Omega,\omega_1)} \\
&\leq \left(||f_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||u||^{p-1}_{W_0^{1,p}(\Omega,\omega_1)}\right)||v||_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and

$$\begin{aligned}
|\Gamma_2(u,v)| &\leq \int_\Omega |b(x,u,\nabla u)||\nabla v|\omega_2 dx \\
&\leq \int_\Omega \left(f_2 + h_2|u|^{q-1} + \tilde{h_2}|\nabla u|^{q-1}\right)|\nabla v|\omega_2 dx \\
&\leq ||f_2||_{L^{q'}(\Omega,\omega_2)}||\nabla v||_{L^q(\Omega,\omega_2)} + ||h_2||_{L^\infty(\Omega)}||u||^{q-1}_{L^q(\Omega,\omega_2)}||\nabla v||_{L^q(\Omega,\omega_2)} \\
&\qquad + ||\tilde{h_2}||_{L^\infty(\Omega)}||\nabla u||^{q-1}_{L^q(\Omega,\omega_2)}||\nabla v||_{L^q(\Omega,\omega_2)} \\
&\leq \left[C_{p,q}||f_2||_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\left(\theta^{q-1}||h_2||_{L^\infty(\Omega)} + ||\tilde{h_2}||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W_0^{1,p}(\Omega,\omega_1)}\right] \\
&\quad ||v||_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and by **(A5)**, Hölder inequality, and Remark 1 **(ii)** , we have

$$\begin{aligned}
|\Gamma_3(u,v)| &\leq \int_\Omega g\ \omega_3^{\frac{1}{p}}\ |u|\ \omega_3^{\frac{1}{s}}\ |v|\ \omega_3^{\frac{1}{s}} dx \\
&\leq \|g\|_{L^p(\Omega,\omega_3)}\|u\|_{L^s(\Omega,\omega_3)}\|v\|_{L^s(\Omega,\omega_3)} \\
&\leq C^2_{p,s}\|g\|_{L^p(\Omega,\omega_3)}\|u\|_{L^p(\Omega,\omega_1)}\|v\|_{L^p(\Omega,\omega_1)} \\
&\leq \theta^2 C^2_{p,s}\|g\|_{L^p(\Omega,\omega_3)}\|u\|_{W_0^{1,p}(\Omega,\omega_1)}\|v\|_{W_0^{1,p}(\Omega,\omega_1)}
\end{aligned}$$

Hence, in (3) we obtain, for all $u, v \in W_0^{1,p}(\Omega,\omega_1)$,

$$\begin{aligned}
|\Gamma(u,v)| &\leq \Big[||f_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||u||^{p-1}_{W_0^{1,p}(\Omega,\omega_1)} + \theta^2 C^2_{p,s}\|g\|_{L^p(\Omega,\omega_3)}\|u\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&\quad + \ C_{p,q}||f_2||_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\left(\theta^{q-1}||h_2||_{L^\infty(\Omega)} + ||\tilde{h_2}||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W_0^{1,p}(\Omega,\omega_1)}\Big] \\
&\quad \|v\|_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

Then $\Gamma(u,.)$ is linear and continuous, for each $u \in W_0^{1,p}(\Omega,\omega_1)$. Thus, there exists a linear and continuous operator on $W_0^{1,p}(\Omega,\omega_1)$ denoted by $\mathcal{A}$ such that

$$\langle \mathcal{A}u, v\rangle = \Gamma(u,v), \quad \text{for all } u, v \in W_0^{1,p}(\Omega,\omega_1).$$

Moreover, we have

$$\begin{aligned}\|\mathcal{A}u\|_* &\leq ||f_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||u||^{p-1}_{W_0^{1,p}(\Omega,\omega_1)} + \theta^2 C_{p,s}^2 \|g\|_{L^p(\Omega,\omega_3)}\|u\|_{W_0^{1,p}(\Omega,\omega_1)} \\ &+ C_{p,q}||f_2||_{L^{q'}(\Omega,\omega_2)} + C_{p,q}^q \left(\theta^{q-1}||h_2||_{L^\infty(\Omega)} + ||\tilde{h_2}||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W_0^{1,p}(\Omega,\omega_1)},\end{aligned}$$

with $\|\mathcal{A}u\|_* := \sup\left\{|\langle \mathcal{A}u, v\rangle| = |\Gamma(u,v)| : v \in W_0^{1,p}(\Omega,\omega_1), \|v\|_{W_0^{1,p}(\Omega,\omega_1)} = 1\right\}$ is the norm in $W_0^{-1,p'}(\Omega,\omega_1^{1-p'})$. Hence, we obtain the operator

$$\begin{aligned}\mathcal{A} : W_0^{1,p}(\Omega,\omega_1) &\longrightarrow W_0^{-1,p'}(\Omega,\omega_1^{1-p'}) \\ u &\longmapsto \mathcal{A}u.\end{aligned}$$

However, the Problem (1) is equivalent to the problem

$$\mathcal{A}u = \mathbf{G}, \quad u \in W_0^{1,p}(\Omega,\omega_1).$$

4.2.2 Coercivity of the operator $\mathcal{A}$

This step establishes that the operator $\mathcal{A}$ is coercive. To this purpose, we have

$$\begin{aligned}\langle \mathcal{A}u, u\rangle &= \Gamma(u,u) \\ &= \int_\Omega \langle a(x,\nabla u),\nabla u\rangle \omega_1 dx + \int_\Omega \langle b(x,u,\nabla u),\nabla u\rangle\omega_2 dx + \int_\Omega g\; u^2\; \omega_3 dx,\end{aligned}$$

for each $u \in W_0^{1,p}(\Omega,\omega_1)$. Moreover, from **(A4)**, $g(x) \geqslant 0$ and Theorem 2, we obtain

$$\begin{aligned}\langle \mathcal{A}u, u\rangle &\geq \beta_1 \int_\Omega |\nabla u|^p \omega_1 dx + \beta_2 \int_\Omega |\nabla u|^q \omega_2 dx + \beta_3 \int_\Omega |u|^q \omega_2 dx \\ &\geq \beta_1 \int_\Omega |\nabla u|^p \omega_1 dx \\ &\geq \frac{\beta_1}{C_\Omega^p + 1}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}.\end{aligned}$$

Hence, since $p > 1$, we have

$$\frac{\langle \mathcal{A}u, u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1)}} \longrightarrow +\infty \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1)} \longrightarrow +\infty,$$

that means, $\mathcal{A}$ is coercive.

4.2.3 Monotonicity of the operator $\mathcal{A}$

The operator **A** is strictly monotone. In fact, for all $\xi_1, \xi_2 \in W_0^{1,p}(\Omega, \omega_1)$ with $\xi_1 \neq \xi_2$, we have

$$\begin{aligned}
&\langle \mathcal{A}\xi_1 - \mathcal{A}\xi_2, \xi_1 - \xi_2\rangle \\
&\quad = \Gamma(\xi_1, \xi_1 - \xi_2) - \Gamma(\xi_2, \xi_1 - \xi_2) \\
&\quad = \int_\Omega \langle a(x, \nabla\xi_1), \nabla(\xi_1 - \xi_2)\rangle \omega_1 dx - \int_\Omega \langle a(x, \nabla\xi_2), \nabla(\xi_1 - \xi_2)\rangle \omega_1 dx \\
&\quad\quad + \int_\Omega \langle b(x, \xi_1, \nabla\xi_1), \nabla(\xi_1 - \xi_2)\rangle \omega_2 dx - \int_\Omega \langle b(x, \xi_2, \nabla\xi_2), \nabla(\xi_1 - \xi_2)\rangle \omega_2 dx \\
&\quad\quad + \int_\Omega g\ \xi_1\ (\xi_1 - \xi_2)\ \omega_3 dx - \int_\Omega g\ \xi_2\ (\xi_1 - \xi_2)\ \omega_3 dx \\
&\quad = \int_\Omega \langle a(x, \nabla\xi_1) - a(x, \nabla\xi_2), \nabla(\xi_1 - \xi_2)\rangle \omega_1 dx + \int_\Omega \langle b(x, \xi_1, \nabla\xi_1) - b(x, \xi_2, \nabla\xi_2), \nabla(\xi_1 - \xi_2)\rangle \omega_2 dx \\
&\quad\quad + \int_\Omega g\ \ (\xi_1 - \xi_2)^2\ \ \omega_3 dx.
\end{aligned}$$

However, thanks to **(A3)** and $g(x) \geqslant 0$, we obtain

$$\langle \mathcal{A}\xi_1 - \mathcal{A}\xi_2, \xi_1 - \xi_2\rangle \quad \geq \quad \int_\Omega \alpha |\nabla(\xi_1 - \xi_2)|^p \omega_1 dx = \alpha \|\nabla(\xi_1 - \xi_2)\|^p_{L^p(\Omega, \omega_1)}.$$

Hence, by Theorem 2, we conclude that

$$\langle \mathcal{A}\xi_1 - \mathcal{A}\xi_2, \xi_1 - \xi_2\rangle \quad \geq \quad \frac{\alpha}{(\theta^p + 1)} \|\xi_1 - \xi_2\|^p_{W_0^{1,p}(\Omega, \omega_1)} > 0.$$

4.2.4 Continuity of the operator $\mathcal{A}$

The operator $\mathcal{A}$ must be shown to be continuous. To this purpose let $u_k \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1)$ as $k \longrightarrow \infty$. Note that if $u_k \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1)$, then $u_k \longrightarrow u$ in $L^p(\Omega, \omega_1)$ et $\nabla u_k \longrightarrow \nabla u$ in $(L^p(\Omega, \omega_1))^n$. Hence, thanks to Theorem 1, there exist (u_{k_j}), $\psi_1 \in L^p(\Omega, \omega_1)$ and $\psi_2 \in L^p(\Omega, \omega_1)$ such that

$$\begin{aligned}
&u_{k_j}(x) \longrightarrow u(x), && \omega_1 - a.e. \text{ in } \Omega \\
&|u_{k_j}(x)| \leq \psi_1(x), && \omega_1 - a.e. \text{ in } \Omega \\
&\nabla u_{k_j}(x) \longrightarrow \nabla u(x), && \omega_1 - a.e. \text{ in } \Omega \\
&|\nabla u_{k_j}(x)| \leq \psi_2(x), && \omega_1 - a.e. \text{ in } \Omega.
\end{aligned} \tag{4}$$

We will show that $\mathcal{A}u_k \longrightarrow \mathcal{A}u$ in $W_0^{-1,p'}(\Omega, \omega_1^{1-p'})$. In order to prove this convergence we proceed in 2 steps.

Step 1:

For $i = 1, ..., n$, we define the operator

$$B_i : W_0^{1,p}(\Omega, \omega_1) \longrightarrow L^{p'}(\Omega, \omega_1)$$
$$(B_i u)(x) = a_i(x, \nabla u(x)).$$

We now show that

$$B_i u_m \longrightarrow B_i u \quad \text{in} \quad L^{p'}(\Omega, \omega_1).$$

The Lebesgue's theorem will be applied.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1)$. Using **(A2)** and Theorem 2(with $\nu = 1$), we obtain

$$\begin{aligned}
\|B_i u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |B_i u(x)|^{p'} \omega_1 dx = \int_\Omega |a_i(x, \nabla u)|^{p'} \omega_1 dx \\
&\leq \int_\Omega \left(f_1 + h_1 |\nabla u|^{p-1}\right)^{p'} \omega_1 dx \\
&\leq C_p \int_\Omega \left(f_1^{p'} + h_1^{p'} |\nabla u|^p\right) \omega_1 dx \\
&\leq C_p \left[\|f_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|\nabla u\|^p_{L^p(\Omega,\omega_1)}\right] \\
&\leq C_p \left[\|f_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}\right].
\end{aligned}$$

(ii) Let $u_k \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1)$ as $k \longrightarrow \infty$.

By **(A2)** and (4), we obtain

$$\begin{aligned}
\|B_i u_{k_j} - B_i u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |B_i u_{k_j}(x) - B_i u(x)|^{p'} \omega_1 dx \\
&\leq \int_\Omega \left(|a_i(x, \nabla u_{k_j})| + |a_i(x, \nabla u)|\right)^{p'} \omega_1 dx \\
&\leq C_p \int_\Omega \left(|a_i(x, \nabla u_{k_j})|^{p'} + |a_i(x, \nabla u)|^{p'}\right) \omega_1 dx \\
&\leq C_p \int_\Omega \left[\left(f_1 + h_1 |\nabla u_{k_j}|^{p-1}\right)^{p'} + \left(f_1 + h_1 |\nabla u|^{p-1}\right)^{p'}\right] \omega_1 dx \\
&\leq C_p \int_\Omega \left[\left(f_1 + h_1 \psi_2^{p-1}\right)^{p'} + \left(f_1 + h_1 \psi_2^{p-1}\right)^{p'}\right] \omega_1 dx \\
&\leq 2 C_p C_p' \int_\Omega \left(f_1^{p'} + h_1^{p'} \psi_2^p\right) \omega_1 dx \\
&\leq 2 C_p C_p' \left[\|f_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|\psi_2\|^p_{L^p(\Omega,\omega_1)}\right].
\end{aligned}$$

Hence, thanks to **(A1)**, we get, as $k \longrightarrow \infty$

$$B_i u_{k_j}(x) = a_i(x, \nabla u_{k_j}(x)) \longrightarrow a_i(x, \nabla u(x)) = B_i u(x).$$

Hence, by Lebesgue's theorem, we get

$$\|B_i u_{k_j} - B_i u\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$B_i u_{k_j} \longrightarrow B_i u \quad \text{in} \quad L^{p'}(\Omega, \omega_1).$$

Finally, we have

$$B_i u_m \longrightarrow B_i u \quad \text{in} \quad L^{p'}(\Omega, \omega_1). \tag{5}$$

Step 2:

We define the operator, for $i = 1, ..., n$, as follows

$$\begin{aligned} &G_i : W_0^{1,p}(\Omega, \omega_1) \longrightarrow L^{q'}(\Omega, \omega_2) \\ &(G_i u)(x) = b_i(x, u(x), \nabla u(x)). \end{aligned}$$

We've also got

$$G_i u_m \longrightarrow G_i u \quad \text{in} \quad L^{q'}(\Omega, \omega_2).$$

In fact,

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1)$. Using **(A2)**, Remark 1 **(i)** and Theorem 2 (with $\nu = 1$), we obtain

$$\begin{aligned} \|G_i u\|^{q'}_{L^{q'}(\Omega,\omega_2)} &= \int_\Omega |b_i(x, u, \nabla u)|^{q'} \omega_2 dx \\ &\le \int_\Omega \left(f_2 + h_2 |u|^{q-1} + \tilde{h_2} |\nabla u|^{q-1} \right)^{q'} \omega_2 dx \\ &\le C_q \int_\Omega \left[f_2^{q'} + h_2^{q'} |u|^q + \tilde{h_2}^{q'} |\nabla u|^q \right] \omega_2 dx \\ &\le C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)} \|u\|^q_{L^q(\Omega,\omega_2)} + \|\tilde{h_2}\|^{q'}_{L^\infty(\Omega)} \|\nabla u\|^q_{L^q(\Omega,\omega_2)} \right] \\ &\le C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q} \left(\theta^q \|h_2\|^{q'}_{L^\infty(\Omega)} + \|\tilde{h_2}\|^{q'}_{L^\infty(\Omega)} \right) \|u\|^q_{W_0^{1,p}(\Omega,\omega_1)} \right]. \end{aligned}$$

(ii) Let $u_k \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1)$ as $k \longrightarrow \infty$.

According to **(A2)**, Remark 1 **(i)** and the same arguments as in Step 1 **(ii)**, we get similarly

$$G_i u_m \longrightarrow G_i u \quad \text{in} \quad L^{q'}(\Omega, \omega_2). \tag{6}$$

Finally, let $v \in W_0^{1,p}(\Omega, \omega_1)$ and using Hölder inequality, Theorem 2 (with $\nu = 1$) and Remark 1, we obtain

$$\begin{aligned} |\Gamma_1(u_k, v) - \Gamma_1(u, v)| &= \left| \int_\Omega \langle a(x, \nabla u_k) - a(x, \nabla u), \nabla v \rangle \omega_1 dx \right| \\ &\le \sum_{i=1}^n \int_\Omega |a_i(x, \nabla u_k) - a_i(x, \nabla u)| |D_j v| \omega_1 dx \\ &= \sum_{i=1}^n \int_\Omega |B_i u_k - B_i u| |D_j v| \omega_1 dx \\ &\le \sum_{i=1}^n \|B_i u_k - B_i u\|_{L^{p'}(\Omega,\omega_1)} \|D_j v\|_{L^p(\Omega,\omega_1)} \\ &\le \left(\sum_{i=1}^n \|B_i u_k - B_i u\|_{L^{p'}(\Omega,\omega_1)} \right) \|v\|_{W_0^{1,p}(\Omega,\omega_1)}, \end{aligned}$$

$$\begin{aligned}
|\Gamma_2(u_k,v)-\Gamma_2(u,v)| &= \Big|\int_\Omega \langle b(x,u_k,\nabla u_k)-b(x,u,\nabla u),\nabla v\rangle \omega_2 dx\Big| \\
&\le \sum_{i=1}^n \int_\Omega |b_i(x,u_k,\nabla u_k)-b_i(x,u,\nabla u)||D_j v|\omega_2 dx \\
&= \sum_{i=1}^n \int_\Omega |G_i u_k - G_i u||D_j v|\omega_2 dx \\
&\le \left(\sum_{i=1}^n \|G_i u_k - G_i u\|_{L^{q'}(\Omega,\omega_2)}\right)\|\nabla v\|_{L^q(\Omega,\omega_2)} \\
&\le C_{p,q}\left(\sum_{i=1}^n \|G_i u_k - G_i u\|_{L^{q'}(\Omega,\omega_2)}\right)\|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\le C_{p,q}\left(\sum_{i=1}^n \|G_i u_k - G_i u\|_{L^{q'}(\Omega,\omega_2)}\right)\|v\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and

$$\begin{aligned}
|\Gamma_3(u_k,v)-\Gamma_3(u,v)| &\le \int_\Omega |g||u_k-u||v|\omega_3 dx \\
&\le \|g\|_{L^p(\Omega,\omega_3)}\|u_k-u\|_{L^s(\Omega,\omega_3)}\|v\|_{L^s(\Omega,\omega_3)} \\
&\le C_{p,s}^2\|g\|_{L^p(\Omega,\omega_3)}\|u_k-u\|_{L^p(\Omega,\omega_1)}\|v\|_{L^p(\Omega,\omega_1)} \\
&\le C_\Omega^2 C_{p,s}^2\|g\|_{L^p(\Omega,\omega_3)}\|u_k-u\|_{W_0^{1,p}(\Omega,\omega_1)}\|v\|_{L^p(\Omega,\omega_1)}.
\end{aligned}$$

Hence, for all $v \in W_0^{1,p}(\Omega,\omega_1)$, we have

$$\begin{aligned}
|\Gamma(u_k,v)-\Gamma(u,v)| &\le |\Gamma_1(u_k,v)-\Gamma_1(u,v)|+|\Gamma_2(u_k,v)-\Gamma_2(u,v)|+|\Gamma_3(u_k,v)-\Gamma_3(u,v)| \\
&\le \Big[\sum_{i=1}^n\Big(\|B_i u_k - B_i u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|G_i u_k - G_i u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\
&\quad + C_\Omega^2 C_{p,s}^2\|g\|_{L^p(\Omega,\omega_3)}\|u_k-u\|_{W_0^{1,p}(\Omega,\omega_1)}\Big]\|v\|_{L^p(\Omega,\omega_1)}
\end{aligned}$$

Then, we get

$$\begin{aligned}
\|\mathcal{A}u_k - \mathcal{A}u\|_* &\le \sum_{i=1}^n\Big(\|B_i u_k - B_i u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|G_i u_k - G_i u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\
&\quad + C_\Omega^2 C_{p,s}^2\|g\|_{L^p(\Omega,\omega_3)}\|u_k-u\|_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

According to (5) and (6), we deduce that

$$\|\mathcal{A}u_k - \mathcal{A}u\|_* \longrightarrow 0 \text{ as } k \longrightarrow \infty,$$

that is, $\mathcal{A}$ is continuous.

Therefore, by Theorem 3, the operator equation $\mathcal{A}u = \mathbf{G}$ admits exactly one solution $u \in W_0^{1,p}(\Omega,\omega_1)$ and it is the unique solution for Problem (1).

4.2.5 Estimates for $\|u\|_{W_0^{1,p}(\Omega,\omega_1)}$

By setting $v = u$ in Definition 3, we get

$$\Gamma(u,u) = \Gamma_1(u,u) + \Gamma_2(u,u) + \Gamma_3(u,u) = \mathbf{G}(u). \tag{7}$$

Onthe other hand, using **(A4)**, $g(x) \geqslant 0$ and Theorem 2(with $\nu = 1$), we obtain

$$\Gamma(u,v) \geq \gamma\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}, \tag{8}$$

where $\gamma = \frac{\beta_1}{\theta^p+1}$.

Next, applying Hölder inequality and Theorem 2(with $\nu = 1$), we get

$$\mathbf{G}(u) \leq |\mathbf{G}(u)| \leq M\|u\|_{W_0^{1,p}(\Omega,\omega_1)}, \tag{9}$$

where $M = \theta||f/\omega_1||_{L^{p'}(\Omega,\omega_1)}$.

According to (7), (8) and (9), we deduce that

$$\gamma\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)} \leq M\|u\|_{W_0^{1,p}(\Omega,\omega_1)},$$

Then

$$\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1)} \leq \frac{M}{\gamma}.$$

Therefore

$$\begin{aligned}\|u\|_{W_0^{1,p}(\Omega,\omega_1)} &\leq \left[\frac{M}{\gamma}\right]^{1/p-1}\\ &= C\left(||f/\omega_1||_{L^{p'}(\Omega,\omega_1)}\right)^{1/p-1},\end{aligned}$$

where $C = \left[\frac{\beta_1}{\theta^{p+1}+\theta}\right]^{1/1-p}$.

As a conclusion, the proof of Theorem 4 is complete.

5 Example

In this section we give an example to illustrate the usefulness of our main results.

Let $\Omega = \{(\xi,\gamma) \in \mathbb{R}^2 : \xi^2+\gamma^2 < 1\}$, and consider the weight functions $\omega_1(\xi,\gamma) = \left(\xi^2+\gamma^2\right)^{-1/2}$, $\omega_2(\xi,\gamma) = \left(\xi^2+\gamma^2\right)^{-3/2}$ and $\omega_3(\xi,\gamma) = \left(\xi^2+\gamma^2\right)^{-1/3}$ $\Big($ note that $\omega_1,\ \omega_2,\ \omega_3 \in A_4,\ p=4,\ q=3$ and $s = 8/3\Big)$. We also define the functions $b : \Omega\times\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $a : \Omega\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g : \Omega \longrightarrow \mathbb{R}$, as follows:

$$a\Big((\xi,\gamma),\eta\Big) = h_1(\xi,\gamma)|\eta|^2 sgn(\eta)\eta,$$

with $h_1(\xi,\gamma) = e^{(\xi^2+\gamma^2)}$, and

$$b\Big((\xi,\gamma),\sigma,\eta\Big) = \tilde{h_2}(\xi,\gamma)|\eta|^2 sgn(\eta),$$

with $\tilde{h_2}(\xi,\gamma) = 2 + sin(\xi^2+\gamma^2)$, and

$$g(\xi,\gamma) = 2 - cos^2(\xi\gamma).$$

Consider the following problem

$$\begin{cases} \mathcal{L}u(\xi,\gamma) = \frac{sin(\xi+\gamma)}{(\xi^2+\gamma^2)^{1/6}} & \text{in } \Omega, \\ u(\xi,\gamma) = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where

$$\mathcal{L}u(\xi,\gamma) = \operatorname{div}\Big[\omega_1(\xi,\gamma)a\Big((\xi,\gamma),\nabla u(\xi,\gamma)\Big)+\omega_2(\xi,\gamma)b\Big((\xi,y),u(\xi,\gamma),\nabla u(\xi,\gamma)\Big)\Big]+\omega_3(\xi,\gamma)g(\xi,\gamma)u(\xi,\gamma). \tag{11}$$

As a result of Theorem 4, the problem (10) has a unique solution $u \in W_0^{1,4}(\Omega,\omega_1)$.

6 Conclusion

In this paper, we studied the existence and uniqueness of weak positive solution for a class of nonlinear degenerate elliptic equations With weight and L^1 data by adopting Sobolev spaces with weight $W_0^{1,p}(\Omega,\omega_1)$. First, we transformed the problem into an equivalent operator equation; second, we utilized the Browder-Minty Theorem to prove the existence and uniqueness of weak solution. We hope in a future work to solve other similar problems by generalization of (1), by replacing, for example, $W_0^{1,p}(\Omega,\omega_1)$ by the weighted Sobolev spaces with variable exponents and other spaces.

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