Finite p-Groups With Noninner Automorphisms of Order p

S.A. Adebisi

Faculty of Science, Department of Mathematics, University of Lagos, Akoka, Yaba, Nigeria
Corresponding author: adesinasunday@yahoo.com

Abstract

Suppose that G is a non-abelian p-group, it was shown that if G is of class 2 then, there exists a noninner automorphism of order p such that $C_G(Z(\Phi(G))) = \Phi(G)$ [1]. Moreover, if G is of maximal class of order p^n, Fouladi S. [13] showed that the order of the group of all automorphisms of G centralizes the Frattini quotient and is not greater than $p^{2(n-2)}$ if and only if G is metabelian. In this paper, we show that if $b(G) = p^2$ and $p \neq 2$, then $\bigcap\{ker\chi \mid \chi(1) = p^2\}$ = 1. (Here, b(G) = max(cd(G)) and cd(G)= $\{\chi(1) \mid \chi \in Irr(G)\}$). Suppose further that G is a p-group with Frattini factor group of order $\geq p^{2a-1}$ we show that the number of elements of order p in G is congruent to 1 modulo p^a $1 \leq a \in \mathbb{N}$.

Keywords: Finite p-group, Automorphism group, Maximal class, Nilpotent groups Noninner automorphisms, p-Groups of class 2, Metabelian group.

1 Introduction

As a result of the recent modern developments in the concepts of finite p-groups, there exists a conjecture of which many contributions have been made through various personalities such as W.Gaschütz [5], M. Deaconescu and G. Silberberg [4], A. Abdollahi [1], and a host of others. The conjecture (see [9]) establishes the fact that G has a noninner automorphism of order p [10]. Also, by a cohomological result of P. Schmid, G admits a noninner automorphism of order p whenever G is regular . Furthermore, a number of studies of the automorphism groups of p-groups of maximal class have been made. For instance , Juhász and Malinowska (see [8] and [10]) concentrate mostly on small automorphism groups. Large automorphism groups were considered in appreciable extents by Shirin Fouladi[13], who was able to show that if G is non-cyclic and of maximal class and of order p^n, then $\mid Aut^{\Phi}(G) \mid = p^{2(n-2)}$ iff G is a metabelian group. Considering the concepts from the character point of view ,observation is made for G that if it is of class 2 and suppose that b(G) = p^2 and $p \neq 2$. By definition, if χ is a character of G , then $ker(\chi) = \{x \in G \mid \chi(x) = \chi(1)\}$ is the kernel of a character χ, where $\chi(1)$ is the degree of a character χ of G. Then, the intersection of the $ker(\chi)$ of which the degree equals p^2 is trivial. We also consider G with Frattini factor group of order not less than $p^{2a-1}, a \in \mathbb{N}$.

2 Statement of Main Result

(a) If G is of class 2, and suppose that b(G) = p^2 . If $p \neq 2$ then , $\bigcap\{ker\chi \mid \chi(1) = p^2\} = 1$, where b(G) = max(cd(G)) and cd(G) = $\{\chi(1) \mid \chi \in Irr(G)$.

(b) Suppose that G is a p-group with Frattini factor group of order $\geq p^{2a-1}$ $a \in \mathbb{Z}$. Then the number of elements of order p in G is $\equiv 1 \bmod p^a$.

Definitions:

(a) Frattini subgroup is the intersection of all maximal subgroups of G. This is denoted by $\Phi(G)$.

(b) A group G is of maximal class if $\mid G \mid \leq p^n, n \geq 3$ and $G = G_0 \geq G_1 \geq \cdots G_n \geq G_{n+1} = \{e\}$. Then, we say that G is of class n and write cl(G) = n > 1.

(c) A group G is metabelian if the quotient group $G/Z(G)$, is abelian. This implies that the commutator subgroup G' is contained in $Z(G)$. Such group possesses a normal subgroup N such that N and G/N are both abelian. The following are metabelian;

   (i) All abelian groups.

   (ii) All generalized dihedral groups.

(iii) All groups of order less than 24.

(iv) All metacyclic groups.

(d) If G is nilpotent of class 2 then, every commutator $[x, y]$ of G, lies in the centre of G, i.e for any $x, y \in G$, $[x, y]$ commutes with any $z \in G$.

Theorem 1: Every finite non-abelian p-group G has a noninner automorphism of order p leaving either the Frattini subgroup $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise fixed, i.e.$C_G(Z(\Phi(G))) = \Phi(G)$ if G is of class 2. The following remarks were applied in the proof of theorem 1.

(1) Suppose that G is a group. Suppose further thatG' the commutator subgroup of G is a finite cyclic p-group for some prime p, then G' is generated by $[x, y]$ for some $x, y \in G$. Now, G' is abelian and the orders of$x, y \in G$ are powers of p, then G' has an exponent which is given by: $max| [x, y] |: x, y \in G$. By the virtue of the fact that G' is a finite cyclic group, it implies that $exp(G') =| G' |$. Clearly, one of the elements of the set $[x, y] : x, y \in G$ is the generator of G'.

(2) From (1), we have that $G' =< [x, y] >, x, y \in G$. Now, for G to be finite and nilpotent of class 2, we have that $G =< x, y > C_G(< x, y >)$ since every commutator $[x, y]$ lies in $Z(G)$ [1]. Observe that for any $g \in G$, $[x, g] = [x, y]^u$ and $[y, g] = [x, y]^v$, where u and v are integers. Here, we have that $[x, y^{-u}x^v g] = 1$ and $[y, y^{-u}x^v g] = 1$. This is because $[x, x^v] = [y, y^{-u}] = 1$. And so, $y^{-u}x^v g \in C_G(< x, y >) \Rightarrow G =< x, y > C_G(< x, y >)$.

(3) Suppose that G is a nilpotent group of class 2, and $a, b \in G \ni 0 < k \in \mathbb{Z}$ Observe that $[a, b]$ lies in the center of G. Then $(ab)^k = a^k b^k [b, a]^{\frac{1}{2}k(k-1)}$ and $[a, b]^m = [a^m, b] = [a, b^m]\ \forall\ m \in \mathbb{Z}$. By the descriptions given above, if $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ is $\ni| [a, b] |= 2^n$ and $a^{m2^n} = b^{-2^n}$, then we have that: $(a^m b)^{2^n} = a^{m2^n} b^{2^n} [b, a^m]^{\frac{2^n(2^n-1)}{2}} = a^{m2^n} b^{2^n} [b, a^m]^{2^{n-1}(2^n-1)} = [b, a]^{m2^{n-1}(2^n-1)}$. Also, note that $[a, b] = [a, a^m b]$, and since $| [a, b] |= 2^n$, then $(a^m b)^{2^{n-1}} \neq 1$.Thus, 2^n divides the order of $(a^m b)$. Therefore , $| a^m b |= 2^{n+1}$, for m $\in \{2r - 1\}_{r\in\mathbb{N}}$, and $| a^m b |= 2^n$, for $m \in \{2r\}_{r\in\mathbb{N}}$.

(4) Suppose that G is a finite p-group of class 2. If G does not have noninner automorphism of order p and $C_G(Z(\Phi(G))) \neq \Phi(G)$. Then, $Z(G)$ must be cyclic, and by implication,

the derived group G' is cyclic.If Z(G) is not cyclic, then $\Omega_1(Z(G))$ is not cyclic and so $\Omega_1(Z(G)) \nleq G'$. Now, if an element t $\in \Omega_1(Z(G)) - G'$ a maximal subgroup R of G and h $\in G - R'$. Then, the map β on G defined by: $(rh^d)^\beta = rh^d t^d \ \forall \ r \in R$ and $d \in \mathbb{N}$, is a noninner automorphism of order p which leaves R (-a resemblance of $\Phi(G)$) elementwise fixed. This is contradictory.

(5) Let G_1, G_2 be subgroups of G $\ni G = G_1G_2$ and $[G_1, G_2] = 1$. If $\exists$ a noninner automorphism $\alpha \in Aut(G_1)$, $\mid \alpha \mid= p$ leaving $Z(G_1)$ elementwise fixed then the map θ on G defined by $(g_1g_2)^\theta = g_1^\alpha g_2 \ \forall \ g_1 \in G_1$ and $g_2 \in G_2$ is a noninner automorphism of G of order p leaving $Z(G)$ elementwise fixed. By the hypothesis, $q^\alpha = q \ \forall$ q $\in G_1 \cap G_2 = Z(G_1)$. Thus, θ is a well defined mapping.

3 Proof of Theorem 1

Suppose that $C_G(Z(\Phi(G))) = \Phi(G)$ and $p = 2$. By Remark 4, let $Z(G)$ be cyclic Then, the implication of Remark 2 is that $\exists \ x, y \in$ G $\ni G' = \langle [x, y] \rangle$. Let $G' = \langle x, y \rangle$ Then, by remark 2, $G = G_1C_G(G_1)$. Also, by remark 5, it is possible to construct a non inner automorphism α of G_1 of order 2 which leaves $Z(G_1)$ elementwise fixed. Observe that $\mid G' \mid=\mid G_1' \mid=\mid [x, y] \mid= 2^n$ for some integer n > 0.

Now, since G' is cyclic and $G' \leq Z(G)$, $exp(G/Z(G)) = exp(G_1/Z(G_1)) = 2^n \Rightarrow Z(G_1) = \langle x^{2^n}, y^{2^y}, [x, y] \rangle \leq Z(G)$. If $n = 1$, then $\Phi(G) = G^2 \leq Z(G)$. By this $C_G(Z(\Phi(G))) = \Phi(G) \Rightarrow G = \Phi(G)$. This is not possible. Thus, $n \geq 2$. $Z(G_1)$ is cyclic. Thus, either $x^{2^n k} = y^{2^n}$ or $x^{2^n} = y^{2^n k}$ for some integer k. Suppose that $x^{2^n k} = y^{2^n}$. If $k \in \{2d\}_{d \in \mathbb{N}}$ then $\mid x^{-k}y \mid= 2^n$ and $x^(-ky)^{2^{n-1}} \notin Z(G_1)$ Then, $\mid [x, y] \mid=\mid [x, x^{-k}y] \mid= 2^n$. Suppose that $v = x^{-k}y$, then the map α on G_1 defined by: $(x^s v^t z)^\alpha = (xv^{2^{n-1}})^s v^t z \ \forall \ z \in Z(G_1)$ and integers s, t, is a noninner automorphism of G_1 of order 2, which leaves $Z(G_1)$ elementwise fixed If $x^{2^n} = y^{2^n}k$ and k $\in \{2d\}_{d \in \mathbb{N}}$, a mapping $\alpha \in Aut(G_1)$. If we assume that $x^{2^n k} = y^{2^n}$ for some integer $k \in \{2d - 1\}_{d \in \mathbb{N}}$. Then, $\mid v \mid=\mid x^{-k}y \mid= 2^{n+1}$. Suppose that $[x, y] \in \langle x^{2^n} \rangle$ Then, $Z(G_1) = \langle x^{2^n} \rangle$. Thus, $\mid x^{2^n} \mid\geq 2^n$. Thus, $x^{2^n i} = v^{2^n}$ for some integer i Now, $\mid x^{2^n} \mid\geq 2^n$, since $n \geq 2$, and $\mid v \mid= 2^{n+1}$, $i \in \{2d\}_{d \in \mathbb{N}}$. By implication, $\mid u \mid=\mid x^{-i}v \mid= 2^n$ and $u^{2^{n-1}} \notin Z(G_1)$ since $\mid [x, y] \mid=\mid [x, u] \mid= 2^n$. Hence, we have the map α on G defined by: $(x^s u^t a)^\alpha = (xu^{2^{n-1}})^s u^t a \forall a \in Z(G_1)$ as the automorphism

α of G_1 required, s,t $\in \mathbb{Z}$. If we assume that $[x, y] \notin \langle x^{2^n} \rangle$, then $\Rightarrow Z(G_1) = \langle [x, y] \rangle = G_1'$, since $Z(G_1) = \langle x^{2^n}, [x, y] \rangle$ is cyclic. Also, consider: $G_1/Z(G_1) = \langle xZ(G_1) \rangle . \langle yZ(G_1) \rangle$ and $| \langle xZ(G_1) \rangle | = | \langle yZ(G_1) \rangle | = 2^n$. This implies that $x^{-2^{n-1}k} y^{2^{n-1}} = \varepsilon \notin Z(G_1)$ and $| \varepsilon | = 2$ as n ≥ 2. The map α on G_1 defined by $(x^s y^t a)^\alpha = (x\varepsilon)^s (y\varepsilon)^t z\ \forall$ z $\in Z(G_1)$ is noninner for s,t $\in \mathbb{N}$.

Theorem 2: Suppose that G is a p-group of maximal class and of order p^n Suppose further that $Aut^\Phi(G)$ is the group of all automorphisms of G which centralizes the Frattini quotient. Then, $| Aut^\Phi(G) | \leq p^{2(n-2)}$ iff G is metabelian.

In order to prove this theorem, it is expedient to consider certain assertions. If G is a p-group of maximal class and of order p^n, let $\Phi = \Phi(G)$ be the Frattini subgroup of G. It has been proved by Satz (see [6]) that the order of $Aut^\Phi(G)$, the group of all automorphisms of G centralizing G/Φ, divides $p^{2(n-2)}$. Let the terms of the lower and upper central series of G be respectively denoted by $L_i(G$ and $U_i(G)$. For n ≥ 4, define the 2-step centralizer C_i in G to be the centralizer in G of $L_i(G)/L_{i+2}(G)$ for $2 \leq i \leq n-2$. Also, define $Q_i = Q_i(G)$ by: $Q_0 =$ G, $Q_1 = C_2$, $Q_i = L_i(G)$ for $2 \leq i \leq$ n. Let the degree of commutativity $\alpha = \alpha(G)$ of G be defined as the maximum integer $\ni$ $[Q_i, Q_j] \leq Q_{i+j+\alpha}\ \forall\, i, j \geq 1$ if Q_1 is not abelian and $\alpha = n-3$ if Q_1 is abelian. Let r $\in G \setminus \bigcup_{i=2}^{n-2} C_i$, $r_1 \in Q_1 - Q_2$ and $r_i = [r_{i-1}, r]$ for $2 \leq ileqn - 1$. Notice that $\{r, r_1\}$ is a generating set for G and $Q_i(G) = \langle r_i, \cdots, r_{n-1} \rangle$, for $1 \leq i \leq n-1$

Theorem A: (see [3]) Let $G = \langle x, y \rangle$ be a 2-generated metabelian group. Then, the following statements are equivalent.

(a) $\forall g_1, g_2 \in G'$, there is an automorphism of G, which maps x to xg_1 and y to yg_2.

(b) G is nilpotent.

Corollary: Suppose that G is a two-group of maximal class and order 2^n. Then, $| Aut^\Phi(G) = 2^{2(n-2)}$. By induction, if n ≤ 4, then G is metabelian and so, $| Aut^\Phi(G) | = p^{2(n-2)}$, by theorem A. On the other hand, suppose that n ≥ 4 and p is odd. Then, we have the following:

Lemma (i): If G is a p-group of maximal class and order p^n. Suppose that: $[Q_2(G), Q_2(G)] \leq Z(G)$, then $[r_i^{-1}, r] = r_{i+1}^{-1}[r_{i+1}, r_i]$ $(i \geq 2)$.

Lemma (ii): Let G be a p-group of maximal class and order p^n, $[Q_{t+1}, Q_{t+1}] = 1$ and $[r_t, r_{t+1}] = w \in$ Z(G) for some t ≥ 2. Then,

(a) $[r_t, r_i] = 1$ for i $\geq$ t + 2,

(b) $[r_{t+1}, r_{t-1}] = [r_{t-1}, r_{t+1}]$,

(c) $[[r_{t-1}, r_t], r] = [r, [t_t, r_{t-1}]]$,

(d) If $[r_{t-1}, r_t] = r_d^{x_d} r_{d+1}^{x_{d+1}} \cdots r_{n-1}^{x_{n-1}}$ for d ≥ t+2 then, $[r_{t-1}, r_{t+1}] = r_{d+1}^{x_d} \cdots r_{n-1}^{x_{n-2}} w^{-1}$.

Lemma (iii): Suppose that G is a p-group of maximal class and of order p^n and $[Q_2(G), Q_2(G)] \leq$ Z(G). Suppose further that the map φ defined by $r^{\varphi} = r$ and $r_1^{\varphi} = r_1 r_2^{-1}$ is an automorphism of G, then $r_i^{\varphi} = r_i r_{i+1}^{-1} [r_{i+1}, r_i]$, i ≥ 2. By induction on G, consider the above corollary for general case of p, and by the metabelian condition, as in lemmas (i), (ii) & (iii), we have that if G is a p-group of maximal class of order p^n. Then, $\mid Aut^{\Phi}(G) \mid = p^{2(n-2)}$ iff G is metabelian, n $\in \mathbb{Z}^+$.

4 Proof of Results (i)

Proposition 1:(Passman(see[7]))If G is of class 2 with b(G) = p^e and e < p then $\bigcap\{ker\chi \mid \chi(1) = p^e\} = 1$.

Proposition 2: Let N $\lhd$ G with G/N a p-group, and $p \neq 2$. Let $C \in Irr(N)$ be invariant in G. Suppose that every irreducible constituent of C^G has degree $\leq p^{C(1)}$. Then, b(G/N) $\leq p$.

Proof: By extending C to $\hat{C} \in Irr(H)$ with N $\subseteq H$ and $\mid G : H \mid = p$, and for $\varphi \in Irr(H/N)$ with $\varphi(1) = p$, consider $(\varphi\hat{C})^G$. $\exists$ linear $\mu \in Irr(H/N) \ni \varphi^x = \varphi\mu$ for x $\in$ G, with μ independent of the choice of φ. And since $p \neq 2$, φ is invariant of G.

By proposition (1), let $C(1) = p$, then C^G is of degree = b(G)$\leq p^2$. Therefore, by the Passman's result, if p is odd, then e < p. The result thus follows. □

Remark: If $p = 2$, then the assertion is actually false.

5 Proof of Results (ii)

Theorem: Let g $\in$ G and let 0 < n $\in \mathbb{N}$. It is possible to find the number of nth roots of g in G. Let $C_n(g) = \mid \{t \in G : t^n = g\} \mid$. If the $gcd(\mid G \mid, n) = 1$, an integer v may be chosen $\ni$ nv $\equiv$ $1 mod \mid G \mid$. Hence, if $t^n = u^n$, then t = $t^{nv} = u^{nv}$ = u (since nv $\equiv 1 mod \mid G \mid$). Thus, $C_n(g) \leq 1$ $\forall$

$g \in G$. Since the map t $\rightarrowtail t^n$ is mono in G, it must also be onto. Thus, we have that $C_n(g) = 1\ \forall$ g $\in G$. Now, by virtue of the fact that C_n is a class function on G, we write

$$C_n = \sum_{\chi \in Irr(G)} V_n(\chi)\chi,$$

where $V_n(\chi)$ a is uniquely determined complex number.

Lemma A:

$$V_n = (\frac{1}{\mid G \mid}) \sum_{g \in G} \chi(g^n)$$

Proof: By the orthogonality relations, we have that:

$$V_n(\chi) = [C_n, \chi] = (\frac{1}{\mid G \mid}) \sum_{g \in G} C_n(g)\overline{\chi(g)}$$

And since

$$C_n(g)\overline{\chi(g)} = \sum_{t \in G : t^n = g} \overline{\chi}(t^n)$$

we have

$$V_n(\chi) = (\frac{1}{\mid G \mid}) \sum_{t \in G} \overline{\chi(t^n)}$$

Replacing t by t^{-1}, we have that

$$V_n = (\frac{1}{\mid G \mid}) \sum_{g \in G} \chi(g^n) \qquad \square$$

Lemma B: Let A $\lhd$ G and $\chi \in Irr(G)$ with A $\subseteq ker\chi$. By definition of $V_n(\chi)$ in lemma A, let $\hat{V}_n(\chi)$ be the corresponding number computed in G/A. Then, $V_n(\chi) = \hat{V}_n(\chi)$.

Proof:

$$\hat{V}_n = (\frac{1}{\mid G : A \mid}) \sum_{Ag \in G/A} \chi((Ag)^n) = (\frac{1}{\mid G : A \mid})(\frac{1}{A}) \sum_{g \in G} \chi(Ag^n) = (\frac{1}{\mid G \mid}) \sum_{g \in G} \chi(g^n) = V_n(\chi). \square$$

Theorem C: (Alperin - Feit - Thompson) (see [7]) Suppose that G is a 2-group containing exactly k involutions. If k $\equiv 1 mod 2^2$, then either G is cyclic or $\mid G : G' \mid = 2^2$.

Lemma D: Let $M \subseteq$ G. Suppose that χ is a character of G. Then, $[\chi_M, \chi_M] \leq\mid G : M \mid [\chi, \chi]$ and $[\chi_M, \chi_M] =\mid G : M \mid [\chi, \chi]$ iff $\chi(g) = 0\ \forall g \in G\setminus M$.

Proof: Consider:

$\mid M \mid [\chi_M, \chi_M] = \sum_{m\in M} \mid \chi(m) \mid^2 \leq \sum_{g\in G} \mid \chi(g) \mid^2 = \mid G \mid [\chi, \chi]$, where $\mid \chi(g) \mid^2 \geq 0$, for $g \in G\setminus M$. Now, if $\chi(g) = 0\ \forall\ g \in$ G $\setminus M$. Then, $\sum_{g\in G} \mid \chi(g) \mid^2 = 0$. Also, since a perfect square is non-negative, this forces $\sum_{m\in M} \mid \chi(m) \mid^2 = 0 \Rightarrow [\chi_m, \chi_m] = \mid G : M \mid [\chi, \chi]$ iff $\chi(g) = 0$. □

Lemma E: Let $\chi \in Irr(G)$ and let c be a linear character of G with $c^n = 1_G$. Then, $V_n(\chi) = V_n(c\chi)$.

Proof: Since $\chi \in Irr(G)$ and c is linear in G with $c^n = 1_G$, then $c\chi \in Irr(G)$. Thus,

$$V_n(c\chi) = (\frac{1}{\mid G \mid})\sum_{g\in G}(c\chi)(g^n) = (\frac{1}{\mid G \mid})\sum_{g} c(g^n)\chi)(g^n) = (\frac{1}{\mid G \mid})\sum \chi(g^n) = V_n(\chi)$$

$$Note\ that : c(g^n) = c(g)^n = c^n(g) = 1. \quad \square$$

Proof of Theorem C: Suppose that G is abelian, then, clearly, it must be cyclic. By induction on G. Suppose that G is not abelian. If Z(G) is not cyclic, choose H $\subseteq$ Z(G), elementary abelian of order 4. Then, the set $\{y \in G \mid y^2 = 1\}$ is a union of cosets of H and hence (k + 1) is a multiple of 4. $\Rightarrow\Leftarrow$. Hence, Z(G) is cyclic and so, G contains the unique minimal subgroup N of order 2. Now since $G' > 1$, then N $\subseteq G'$. Also, G/N is not cyclic since G is nonabelian. If G/N satisfies the hypothesis of the theorem, then $\mid G : G' \mid = \mid (G/N) : (G/N)' \mid = 4$, by induction. Now, assuming that the number of involutions in G/N is not $\equiv 1 mod 2^2$, we have that:

$$\sum_{\chi\in Irr(G)} V_2(\chi)\chi(1) = k + 1 = 2 mod 2^2$$

and

$$\sum_{\chi\in Irr(G):N\subseteq ker\chi} V_2(\chi)\chi(1) not \equiv 2 mod 2^2$$

and so,

$$\sum_{\chi\in Irr(G):N\nsubseteq ker\chi} V_2(\chi)\chi(1) not \equiv 0 mod 2^2 \cdots (*)$$

Suppose that $\mathfrak{L}$ is a group of linear characters c of G which satisfies $c^2 = 1_G$.

Then, for $\chi \in Irr(G)$, we have that $c\chi \in Irr(G)$ and since N $\subseteq ker(c\chi)$.

Hence, $\mathfrak{L}$ permutes $\{\chi \in Irr(G) \mid N \subseteq ker\chi\}$ and partitions this set into orbits $\mathfrak{S}_i$. By lemma E, $V_2(\chi)$ is constant on each orbit as is $\chi(1)$. Now, $\mid \mathfrak{S} \mid$ is a power of 2. From (*) $\exists\ \chi \in Irr(G) \ni$

(i) N $\subseteq ker\chi$

(ii) $V_2(\chi) \neq 0$,

(iii) $\chi(1) \mid \mathfrak{S} \mid \leq 2$.

Taking $\mathfrak{S}$ as the orbit containing χ. Now, since N $\nsubseteq ker\chi$, we have that $ker\chi = 1$. G is not abelian, thus $\chi(1) > 1$ and thus, $\chi(1) = 2$, and $\mid \mathfrak{S} \mid = 1$. This implies that $c\chi = \chi, \forall c \in \mathfrak{L}$. Let T = $\Phi(G)$, the Frattini subgroup. Suppose that g $\in G\backslash$ T. Then *exists* c $\in \mathfrak{L}$ for which c(g) $\neq$ 1 and we have that $\chi(g) = 0$. By lemma D, we have that $\mid G : T \mid = [\chi_T, \chi_T] \leq \chi(1)^2 = 2^2$. Observe that G is noncyclic; $\mid G, T \mid \geq 4$, and so, $4 = \chi(1)^2 = [\chi_Q, \chi_Q] = [G, T]$ Thus, we must have $\chi_T = 2\varphi$, where φ is a faithful linear character of T. Now since $V_2(\chi) \neq 0$, $\chi \in \mathbb{R}$. Hence, φ is also real. Thus, $\mid Q \mid \leq 2 \Rightarrow \mid G \mid \leq 2^3$ and so G $\cong D_{2^3}$ or Q_{2^3}. In either case, $\mid G, G' \mid = 2^2$.□
Applying theorem C, we have that if $\mathfrak{B}$ is the set of the elements of group G $\ni \mathfrak{B} = \{b :\mid b \mid = p\}$, then $\mid \mathfrak{B} \mid \equiv 1 mod p^a$, $a \in \mathbb{N}$.

Remark: Theorem (O. Tausssky)(see [6]). The only nonabelian 2-groups G for which $\mid G, G' \mid = 2^2$ are the dihedral, semidihedral, and generalized quaternion groups, where the number of involutions is congruent to 1 modulo 4.

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