A Novel Predator-Prey Modelling Approach to Survival Rate of Fish in an Unbounded Aquatic Environment

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Abstract

This paper presents a survival analysis of fish in an unbounded aquatic environment where fish, zooplankton and phytoplankton interact freely. we formulate a predator-prey model by incorporating optimal selective harvesting. From the numerical results, it is observed that as the selling price of unit biomass of fish and instantaneous time delayed annual discount rate increases, then the optimal harvesting rate of fish increases gradually. Therefore, the harvesting of fish species will be increased whenever the selling price of fish and the annual discountof fish production cost increases. It is also found that the optimal selective harvesting rate of fish decreases due to the increase on cost of harvesting of fish.

Keywords: predator-prey modelling, survival rate, unbounded aquatic environment.

Introduction

Descriptions of the interactions between species and prediction of the future state of an ecosystem helps to maintain and sustain the benefits that we extract from nature [1]. Developing Mathematical models is one of the key approaches applied in understanding the ecological interaction between predators and prey species [2]. Mathematical modeling and analysis of ecological problems have been used to understand more complex ecological interactions by studying the dynamics of predator-prey systems [3], [4] and [5]. The problem of time delay optimal selective harvesting in predator-prey system is a dominanttheme in ecology and bio-economics due to its importance [6], [7] and [8]. More realistic and plausible mathematical models require critical consideration of aspects such as carrying capacity [9], competition among predators and prey [10],harvesting of prey or predator [11], and functional responses of predators [12]. Anidea about predator interactions is one source of information for strategic management of ecosystems since predation regulate the number of prey and their survival which is a source of potential change in biological environment [13], [14], [15], [16] and [17]. Knowledge of both functional and numerical responses is required to fully understand how predators and prey interact hence providing a complete description of predator population dynamics [18], [19], [20], [21], and [22]. Many species have experienced extinction while others are approaching it due to factors like; poor management of natural resources [23], environmental pollution [24], over-predation [25], over-exploitation [26] among others. To protectthese species from extinction, precautions like creation of reserve zones and restriction on harvesting should be put in place to allow them grow without any external disturbance [27]. The existence of reserve regions also called refuges have become a key interest to researchers in studyingthe predator-prey dynamics. In his work [28], Holling came up with three major types of functional responses namely; types I, II and III and theeffect they have on prey killed per unit time. Holling type II responses are characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited to by its capacity to process food [29].Holling type II response is often modeled by a rectangular hyperbola, for instance, by Holling disc equation which assumes that processing food and searching for food are mutually exclusive behaviors [30]. However, as indicated, the responses (and many more) can be derived from a system offast state transitions of the prey or predator during which the total prey and predator densities remain constant [31]. Mathematical models are usually used to analyze the dynamics of com- plex interacting populations. We note that many researchers have so far greatly studied the relationship that exists among biological species in the past few decades using varying methods [32]. The Lotka-Volterra model is one of the earliest predator-prey models to be based on sound mathematical principles [33]. It's the basis of many models currently being used in analysis of population dynamics. It entails two coupled nonlinear differential equations that show the interaction between a predator and prey population as indicated;

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy+- dxy \qquad (1)$$

In the System 1, the constant a represents prey population net growth rate in the absence of predators and ax is growth term. The constant c represents predator population death rate when prey is absent. Finally, cy is decay term. The xy term represents the interaction between two populations, given that both species move about randomly and are uniformly distributed over their habitat. The derivatives represent the rates of change in both populations with respect to time t [19]. From the above model, a large number of prey population ensures morefood to support a large predator population. Equally, it is importantto note that when the predator population increases, prey begins to die leading to a decrease in the number of predators. A functional response is a key feature in any ecosystem since it describesthe consumption rate of a given prey by a predator. The modified two dimensional predator-prey model also uses a nonlinear system of equations that includes logistic growth of two species, a carrying capacity of the prey, and a predatory factor.

Preliminaries and notations

In this section, we introduce elementary mathematical ideas and definitions that are useful in the sequel.

Definition 2.1. Functional responses

This response describes the relationship between the rate of consumption by a single predator and prey density; implying that the number of preys eaten per predator per unit time, changes with prey densities. The functional responses applied in an ecological modeling is classified into; prey-dependent, predator-dependent and ratio dependent or multi-species dependent. The Types of functional responses include:

Holling type I: It describes a linear increase in the rate of consumption for each individual predator as the number of preys rises up to a maximum point where consumption level becomes constant [1]. It is expressed as, $N = aT_s x$, where $x \geq 0$, N is the number of preys consumed and aT_s is the consumption rate of prey by a predator within a given time.

Holling type II: In this case, the consumption rate of each consumer rises at a reducing rate with prey density until it becomes constant at satiation (saturation level) [13]. In addition to time available for searching, it also takes care of both handling and ingestion time, b for each individual prey that is consumed, hence, the searching time is reduced to the form $T_s = T_t - b$. Now, when this equation is combined with one for Holling type I, we come up with type II formulation given as**:** N = $\frac{aTtx}{1+abx}$.

Holling type III: In this type, Holling proposed functional response of the form, N $=\frac{aTtx^k}{1+abx}$, if $k = 2$, where k is an integer. It is a generalization of type II functional response and it describes those situations in which mortality ofthe prey first increases with low prey densities, later, it decreases at highprey densities so that the response curve has a characteristic S-shaped form. In general terms, the function N = $\frac{x^k}{a+x^k}$. represents a functional response of predator to prey, which is called Hollingtype II if $k = 1$ and Holling type III if $k = 2$.

Definition 2.2. Prey-dependent rate

The consumption rate by each predator which is only a functionof prey $f(x, y) = f(x)$.

Definition 2.3. Ratio dependent

This is a predator dependent response where the functional response only depends on the ratio of prey population size to predator population size.The pace of growth of predators is governed by how they transform consumed prey into new predators [27].

Definition 2.4. Multi-species dependent

Multi-species functional responses are functional reactions that are dependent on the abundances of multiple prey species.

Research Methodology

This section entails the description of methods and techniques which are useful in the analysis of the problem. We consider techniques for stability analysis, simulation and numerical analysis. Accuracy, stability and efficiency are three numerical properties that areused to determine the performance of numerical methods. The error caused by a small perturbation in the numerical method remains bounded [24]. This could happen unconditionally in the entire domain of definitionor conditionally within a range. Numerical methods in getting solutions of ordinary differential equationscan be put in two categories - Numerical integration methods and Runge-

Kutta methods [22]. Studies carried out analytically cannot be complete without verifying the formulated model numerically. Therefore, there is need to carry out simulations of the dynamical behaviour of the system using Runge-Kutta iteration methods discussed in the section above. To carry out this procedure, we choose the values of the parameters following ecological observations which are realistic although they are hypothetical in nature.

Results and Discussion

Model Formulation

Consider a two species prey-predator interaction in which $x(t)$ and $y(t)$ denote the population density of prey species and predator species respectively at any time t. Then, the generalized prey-predator model is given by:

$$\frac{dx}{dt} = \text{ax-xp(x)y},$$
$$\frac{dy}{dt} = \text{-dy}+\propto \text{xp(x)y}, \tag{4.1}$$

where a, d, α and $xp(x)$ represents the specific growth rate of prey population in the absence of predator, natural death rate of predators in the absence of prey, conversion factor and response function respectively. If we then assume that the prey population grows logistically in the absence of predators with a growth rate r and carrying capacity k, then System 4.1 changes to:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)\text{-} xp(x)y,$$
$$\frac{dy}{dt} = \propto xp(x)y\text{-}dy \tag{4.2}$$

Let the functional response function $xp(x)$ be expressed in the form of $xp(x) = \frac{mx}{1+x}$ corresponding to a Holling type II functional response. Then the System 4.2 becomes:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x},$$
$$\frac{dy}{dt} = \propto \frac{mxy}{1+x} - dy. \tag{4.3}$$

If for economical purpose, we only let the predator species be subjected to harvesting, then the System 4.3 changes to:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)-\frac{mxy}{1+x},$$

$$\frac{dy}{dt} = \propto \frac{mxy}{1+x} - dy - q_2E_2y, \tag{4.4}$$

where q_2 is the catchability coefficient and $0 < E_2t < E_{max}$ is the harvesting effort of the predator species. Now, if we introduce a time delay constant, $(\tau \geq 0)$ in the harvesting term, then the System 4.4 extends to:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{1+x},$$
$$\frac{dy}{dt} = \propto \frac{mxy}{1+x} - dy - q_2E_2y(t- \tau), \tag{4.5}$$

which is the Holling type II functional response model with a time delay predator harvesting. Similarly, if we assume that only the prey species are selectively harvested, for their economic value, then System 4.5 can as well be written as:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{1+x} - q_1E_1x,$$
$$\frac{dy}{dt} = \propto \frac{mxy}{1+x} - dy. \tag{4.6}$$

Introducing the time delay constant, $(\tau \geq 0)$ in the harvesting term leads to the require Holling type II response model with only prey harvesting given by:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{1+x} - q_1E_1x(t- \tau),$$
$$\frac{dy}{dt} = \propto \frac{mxy}{1+x} - dy. \tag{4.7}$$

The system 4.5 and System 4.7 are formulated under the following assumptions:
(i). The prey species grow logistically in the absence of predators.
(ii). The predator feeds on the prey according to a Holling type II functional response.
(iii). Prey species find enough food at all times.
(iv). Only one of the species is subjected to harvesting hence selective harvesting.
(v). The catch rate function q_iE_i is based on the catch-per-unit effort.
(vi). Harvesting of species begin to occur after a certain age or size.
The meanings of the parameters used in the formulated models are explained as per the table below.

Parameter	Meaning
$x(t)$	Population density of prey species at time t
$y(t)$	Population density of predator species at time t
r	Intrinsic growth rate of prey species
k	Carrying capacity for prey species
m	Capturing rate of predator on prey
α	Conversion rate of prey to predator
d	Natural death rate of predator in the absence of prey
q_2	Catchability coefficient of predator
q_1	Catchability coefficient of prey
E_1	Harvesting effort of prey
E_2	Harvesting effort of predator
τ	Time delay constant

Numerical simulations

Studies carried out analytically cannot be complete without verifyingthe formulated model numerically using MATLAB software. We there- fore carry out simulations of the dynamical behaviour of the system usingRunge - Kutta iteration methods discussed in Chapter three. We choose the parameters following ecological observations which are realistic al- though they are hypothetical in nature. The parameter values are as follows: $r = 2.05$, $k = 121$, $\beta = 0.59$, $\alpha = 3.98$, $\gamma = 0.48$, $\beta_0 = 0.4$, $d = 0.03598$, $\rho = 0.99$, $s = 0.25$, $s_1 = 0.25$, $\delta = 0.65$, $q = 0.015$, $E = 0.39$, $\gamma_1 = 0.09$.

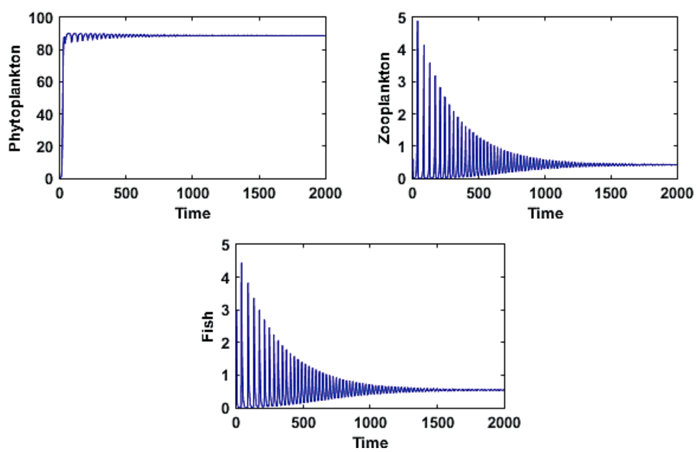

Figure 4.1: Population densities of Phytoplankton, Zooplankton andFish over time evolution.

The hypothetical set of values of the parameters has been used in drawing Figure 4.1. We can see from this figure that we have locally asymptotically stable equilibrium point from the interior illustrating thatPhytoplankton, Zooplankton and Fish species coexist. At this point we consider the numerical analysis of the optimal selective harvesting problem which has been solved by use of the values of theparameters as: $r = 2.15$, $k = 101.5$, $\beta = 0.58$, $\alpha = 1.01$, $\gamma = 0.66$, $\beta_1 = 0.51$, $d = 0.3501$, $\rho = 0.195$, $\gamma_1 = 0.62$, $s = 0.49$, $s_1 = 0.37$, $\delta = 0.019$, $q = 0.019$. We have used the Forward Runge-Kutta method to solve the System 4.5 within a specified time interval. We follow the procedure by the use of Backward Runge-Kutta method to solve the optimal selective harvesting problem in System 4.7 Finally, the optimal selective harvest- ing results are displayed with consideration to selling price of fish (p), the harvesting cost (c) and instantaneous time delayed annual discount rate(δ_1) respectively.

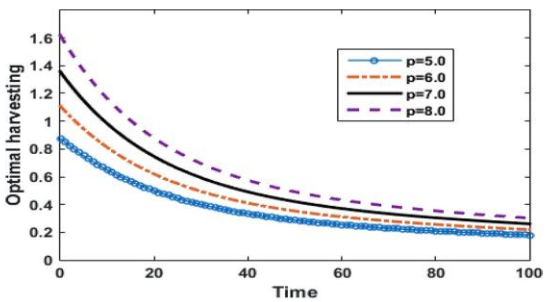

Figure 4.2: Optimal harvesting of Fish species with respect to sellingprice (p).

It is observed from Figure 4.2 that an increase in the selling priceof a unit biomass of fish leads to an increase in the optimal selective harvesting rate of fish species. This increase happens gradually.

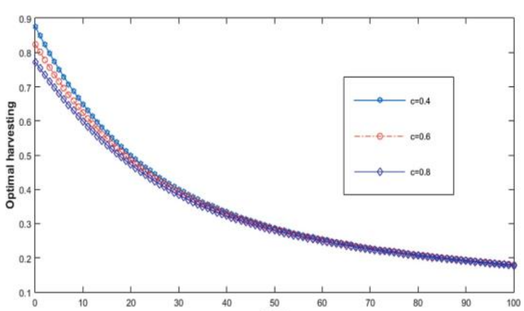

Figure 4.3: Optimal harvesting of Fish species with respect to harvesting cost (c).

From Figure 4.3, it is clear that as the cost of harvesting of fish increases, two things happen. Firstly, the optimal selective harvesting of fish gradually decreases. Secondly, after that gradual decrease it goes tothe equilibrium level.

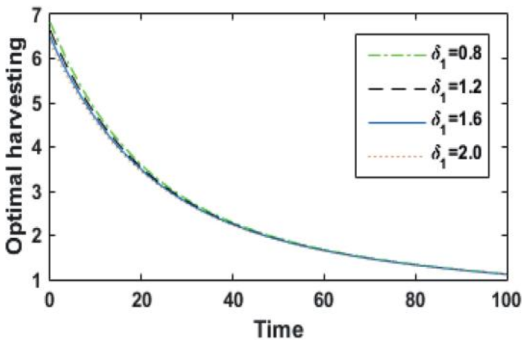

Figure 4.4: Optimal harvesting of Fish species with respect to annualdiscount rate (δ_1) of Fish production cost.

From Figure 4.4, we observe that an increase in the annual time delayed discount rate of selling price leads to an increasein the rate of the optimal selective harvesting of fish. These changes also happen gradually under time delay. Therefore, we conclude that theincrease of time delayed annual discount rate of selling price can increasethe optimal selective harvesting rate of fish.

Conclusion

We have formulated the model in Equations 4.5 and 4.7 by incorporating optimal selective harvesting. We have done both local and global analysis of the model.We have observed that the system continues to have oscillatory behaviour for $\gamma_1 < 0.00604$, but assumes a stable steady state behaviour for $\gamma_1 \geq 0.00604$. Therefore, it can be concluded that the system may become stable for the higher rate of consumption of zooplankton by fish species. Finally, we have done numerical simulations of the model and given the graphical representations and their interpretations. From the numerical simulation results of the optimal selective harvesting problem, it is observed that as the selling price of unit biomass of fish and instantaneous time delayed annual discount rate increases, then the optimal harvesting rate of fish increases gradually. Therefore, the harvesting of fish species will be increased whenever the selling price of fish and the annual discount of fish production cost increases. It is also found that the optimal selective harvesting rate of fish decreases due to the increase on cost of harvesting of fish. Since these results are making practical sense, then this model can be implemented in a fishery system.

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