

Properties of Derivations Induced by Orthogonal Projections

I. O. Okwany, N. B. Okelo*, O. Ongati

Department of Pure and Applied Mathematics,
Jaramogi Oginga Odinga University of Science and Technology,
P. O. Box 210-40601, Bondo-Kenya

*Corresponding author: bnyaare@yahoo.com

Abstract

Let H^n be a finite dimensional Hilbert space and $\delta_{P,Q}$ be a generalized derivation induced by the orthogonal projections P and Q . In this study, we have approximated the norm of $\delta_{P,Q}$ by the formula $\|\delta_{P,Q}\| = \{\sum |\alpha|^2\}^{\frac{1}{2}} + \{\sum |\beta|^2\}^{\frac{1}{2}}$ and also showed that $\delta_{P,Q}$ is bounded and positive $\|\delta_{P,Q}\| \geq 0$ whenever P and Q are positive. Finally, we show compactness of $\delta_{P,Q}$ for compact operators P and Q .

Keywords: Derivation, Orthogonal projection, Compactness, Hilbert space.

Introduction

Studies have been done on generalized derivations, inner derivations, aspects of the underlying algebra $B(H)$ of these derivations and the structures of the operators inducing the derivations. An operator T is called D -symmetric, if the closure of the derivation δ_A is equal to the closure of the derivation δ_{T^*} in the norm topology. Anderson, Deddens and Williams [1] showed that for a trace class operator τ , $TP = PT$ implies that $T^*P = PT^*$ if T is D -symmetric operator. A generalization of this concept was used in [13] to define a class of pairs of operators $A, B \in B(H)$ say, such that $BT = TA$, implies that $T^*P = PT^*$, T^*, P^* being the adjoints of T and B respectively and T an element of trace class operators i.e. P -symmetric operators. Salah [14] constructed different C^* -algebras using the elements of P -symmetric operators i.e. $A, B \in B(H)$ such that $TA = AT$ implies that $A^*T = TB^*$. Indeed by [13], for $A, B \in B(H)$, if the pair (A, B) is generalized P -symmetric then: $\tau_0(A, B)$, $\iota(A, B)$ and $\kappa(A, B)$ are C^* -algebras w^* -closed in $B(H) \times B(H)$ and $\tau(A, B)$ is a bilateral ideal of $\iota(A, B)$. Continuity of derivations as mappings on different algebras is an important concept which has been fairly researched on. Kaplansky [8] and later Sakai [15], proved that a derivation δ of a C^* -algebra is automatically norm-continuous. This idea was later employed by Kadison [6] to show that such derivation is also continuous in the ultra-weak topology only if

such a derivation is of an algebra of operators acting on a Hilbert space. Johnson [5] and later Sinclair [17] proved the automatic norm continuity of derivations of a semi-simple Banach algebra. Ringrose [12] used cohomological notation to prove that derivations from a C^* -algebra into a Banach-Module are automatically norm continuous, and that for appropriate class of dual algebra modules, they are continuous also relative to the ultraweak topology on the algebra and the weak $*$ -topology on the module [12]. A linear mapping on an algebra X into an X -bimodal M is called a local derivation if for each $T \in A$, there is a derivation δ_T of X into M such that $\delta_T = \delta_T(T)$ [7]. Most of the studies on local derivations have been focused on finding the conditions which imply that a local derivation is a derivation. It is shown by Bresar [9] that in certain algebra, derivations can be characterized by some properties which local derivations trivially have, for example; Let X be a von Neumann algebra and let M be a normed X -bimodule. If a norm-continuous linear mapping δ of X into M is a local derivation, then δ is a derivation.

A linear mapping T on a complex unital Banach algebra A is spectrally bounded if $r(Tx) \leq Mr(x)$ for all $x \in X$ and some $M \geq 0$ where $r(\cdot)$ denotes the spectral radius [4]. Bresar [9] affirmed the fact that the image δ_X of an inner derivation δ of X is contained in the radical $radX$ of X if and only if δ is spectrally bounded, where $radX$ is the Jacobson radical. His argument was essentially based on the results due to Ptak [11], that a spectrally bounded inner derivation has the property that $\delta^2X \subseteq \theta(X)$, the set of quasinilpotent elements of X . Curto [4] later on characterized the generalized inner derivations on a unital Banach algebra which are spectrally bounded. In particular,[4] simplified the argument due to [9], that every spectrally bounded inner derivation that maps into the radical is attainable [4]. Suppose $L(X, Y)$ is a space of all linear maps between Banach spaces X and Y , and S is a subset of $L(X, Y)$, a mapping $\Delta: X \mapsto T$ is said to be weak-2-local S map if for every $x, y \in X$ and $\phi \in Y^*$, there exists $T_{x,y,\phi} \in S$, depending on x, y and ϕ satisfying $\phi\Delta(x) = \phi T_{x,y,\phi}(x)$, and $\phi\Delta(y) = \phi T_{x,y,\phi}(y)$. The idea of weak-2-local derivations and automorphisms was introduced by Semrl [16] and explored extensively in [3] and [2]. In [10], Niazi and others proved that every weak-2-local derivation on a finite dimensional C^* -algebra is a linear derivation, and every weak2-local $*$ -derivation on $B(H)$ is a linear $*$ -derivation. It was then proved that every (weak)-2-local derivation on $C_0(L, A)$ is a linear derivation [3]. Consequently, [3] also showed that if B is an atomic von Neumann or a compact C^* -algebra, then every weak-2-local derivation on $C_0(L, B)$ is a linear derivation. Furthermore, for a general von Neumann algebra M , every 2-local derivation on $C_0(L, M)$ is a linear derivation.

We begin by applying the properties of orthogonal projections P and Q to construct a new orthogonal projection $P - Q$. We then proceed to apply these properties to give examples of the same on finite dimensional Hilbert space using matrices. We then construct a derivation $\delta_{P,Q}(X) = PX - XQ$ and show that $\delta_{P,Q}$ is a bounded linear operator which is continuous and positive. Finally, we calculate the norm $\| \delta_{P,Q} \|$ of the derivation $\delta_{P,Q}$ and determine the norm and numerical radii inequalities for the same. In each of the properties of $\delta_{P,Q}$ discussed, we infer the results to the case when $P = Q$ to obtain the result for inner derivation δ_P . We shall denote set of all orthogonal projections acting on a Hilbert space H by $P_0(H)$.

Remark. The set of all derivations induced by orthogonal projections shall be denoted by $D_{op}[B(H)]$. Similarly, we shall denote by $D_{op}^I [B(H)]$ and $D_{op}^G [B(H)]$ respectively the sets of all inner derivations and generalized derivations induced by orthogonal projections. It is noted that if $P = Q$ then $D_{op}^G [B(H)] \Leftrightarrow D_{op}^I [B(H)]$. Let H be a Hilbert space with a decomposition $H = V \oplus W^\perp$ where W^\perp is the orthogonal complement of W . Suppose that $P, Q \in P_0(H)$ are orthogonal projections on V and W respectively, then for any arbitrary linear operator X , there exists a new orthogonal projection $\delta_{PQ}(X) = (PX - XQ) \in D_{op}[B(H)]$ which acts on the subspace $V \oplus W^\perp$.

1. Basic definitions

Definition 2.1 (52, Section 1). Let $B(H)$ be a C^* -algebra of all bounded linear operators on a Hilbert space H . An operator $T_{A,B} : B(H) \mapsto B(H)$ is called an elementary operators if it has the representation $T(X) = \sum_{i=1}^n A_i X B_i, \forall X \in B(H)$ where A_i, B_i are fixed in $B(H)$ or $M(H)$, the multiplier algebra of $B(H)$. For A and B fixed in $B(H)$, for all $X \in B(H)$ we define the particular elementary operators:

- (i). the left multiplication operator (implemented by A) $L_A : B(H) \mapsto B(H)$ is defined by $L_A(X) = AX$.
- (ii). the right multiplication operator (implemented by B) $R_B : B(H) \mapsto B(H)$ is defined by $R_B(X) = XB$.
- (iii). the generalized derivation (implemented by A, B) $\delta_A : B(H) \mapsto B(H)$ is defined by $\delta_{A,B}(X) = AX - XB$.
- (iv). the inner derivation (implemented by A) $\delta_A : B(H) \mapsto B(H)$ is defined by $\delta_A(X) = AX - XA$.
- (v). the basic elementary operator (implemented by A, B) $M_{A,B}(X) = AXB$

(vi). the Jordan elementary operators (implemented by A, B)

$$\mathcal{U}_{A,B}(X) = AXB + BXA, \forall X \in B(H).$$

Definition 2.2. Let $M_n(\mathbb{K})$ be a space of matrices over \mathbb{K} . The norm of $A \in M_n(\mathbb{K})$ is a function defined by $\|A\| = \max\{\|A\bar{v}\| : \|\bar{v}\| = 1\}$ for a vector v which obeys all the norm properties and in addition, it is submultiplicative and subadditive i.e $\|AB\| \leq \|A\| \|B\|$ and $\|A + B\| \leq \|A\| + \|B\|$ for $A, B \in M_n(\mathbb{K})$

Example 2.3. The following are some examples of the matrix (operator) norms:

(i). One-norm (the ℓ^1 -norm) $\|T\|_1 = \sum_{j=1}^n |a_{ij}|$. Let $M_n(\mathbb{R}) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be given by

$$T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \text{ thus for unit vectors } x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ then } T(x_1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } T(x_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $\|T(x_1)\| = \left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\| = |1| + |3| = 4$ and $\|T(x_2)\| = \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\| = |2| + |1| = 3$ therefore

$$\|T\|_1 = 4.$$

(ii). Max-norm (the ℓ^∞ -norm) $\|T\|^\infty = \max |a_{ij}|$. Let T be as given in (i) above and vectors $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then $T(x_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T(x_2) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(iii). Two-norm (the ℓ^2 -norm on T) $\|T\|_2 = (\sum_{j=1}^n |a_{ij}|^2)^{\frac{1}{2}}$.

Definition 2.4 (1, Definition 2.1). Let $T \in B(H)_1 \mapsto B(H)_2$ be a bounded linear operator and H_1, H_2 finite dimensional Hilbert spaces. The norm of the operator T is the smallest real number $\|T\|$ such that $\|Tx\| \leq \|T\| \|x\|$ where, $x \in H_2$, i.e $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$.

Remark 2.5. Given that $T \in B(H)$ is a compact operator, then we denote by $\{s_j(T)\}$, the singular values of T i.e the eigenvalues of $|T| = (TT^*)^{\frac{1}{2}}$. Schatten- p norm is an operator norm defined by $\|T\|_p = (\sum_{j=1}^\infty s_j^p(T))^{\frac{1}{p}}$ for $1 \leq p \leq \infty$. For strictly positive p , the class of operators which admits the norms $\|T\|_p = (\sum_{j=1}^\infty s_j^p(T))^{\frac{1}{p}}$ are called Schatten- p operators and are denoted by C_p . C_p is an ideal in $B(H)$ of compact operators whose $\|T\|_p < \infty$, so that $\| |T|^2 \|_p = \|T\|_p^2$ for a finite p .

The C_p class has two subclasses for $p = 1$ and $p = 2$ given by:

(i). Taxicab norm (C_1): For $p = 1$ then $\|T\|_1 = (\sum_{j=1}^\infty s_j(T))$ and $\| |T|^2 \|_{\frac{1}{2}} = \|T\|_1^2$. The class of all operators which admit the norm $\|T\|_1 = (\sum_{j=1}^\infty s_j(T))$ are called is called Trace class and is denoted by C_1

(ii). Hilbert-Schmidt norm (C_2): For $p = 2$ then $\|T\|_1 = (\sum_{j=1}^\infty s_j(T))^{\frac{1}{2}}$ and $\| |T|^2 \|_{\frac{1}{2}} = \|T\|_1^2$. The class of all operators which admit the norm $\|T\|_1 = (\sum_{j=1}^\infty s_j(T))^{\frac{1}{2}}$ are called is called Trace class and is denoted by C_2

Remark 2.6. The effect of an operator on a vector is a measure of how much an operator amplifies a norm of a unit vector. Operator norm $\|T\|$ is generally a vector norm on the range of the operator

T such that $\|T^2\| \leq \|T\|^2$. An operator acting on a finite dimensional Hilbert space can be represented by a matrix.

Definition 2.7. Suppose that $U, V, T \in B(H)$, where U and V are both unitary and A being compact, then a norm $\| \cdot \|$ defined by $\|UTV\| = \|T\|$ is called unitarily invariant norm.

Definition 2.8. Let H be a complex Hilbert space and T be a linear operator from H to itself. T is said to be positive if $\langle Tx, x \rangle \geq 0$, for all $x \in H$. This is denoted by $T \geq 0$ or $0 \leq T$. T is then said to be strictly positive or positive definite if $\langle Tx, x \rangle > 0$, for all $x \in H \setminus \{0\}$.

2. Results and discussions

We shall denote set of all orthogonal projections acting on a Hilbert space H by $P_0(H)$. In the sequel, we shall consider two decompositions of H^n thus; $H^n = H_1 \oplus H_2$ and

$$H^n = H_{11} \oplus H_{22} \text{ so that } H^n = H_1 \oplus H_2 = H_{11} \oplus H_{22}$$

Lemma 3.0.1. Suppose there exist two distinct ways of decomposing H^n , $H^n = H_1 \oplus H_2$ and $H^n = H_{11} \oplus H_{22}$ and if $H_1 \subset H_{22}$ or $H_{11} \subset H_2$, then: $H^n = (H_1 \oplus H_{11}) \oplus (H_2 \cap H_{22})$:

Proof.

Given that $H_1 \subset H_{22}$, then $H_1 + (H_2 \cap H_{22}) = (H_1 + H_2) \cap H_{22} = H^n \cap H_{22} = H_{22}$: and because $H_1 \cap (H_2 \cap H_{22}) = (H_1 \cap H_2) \cap H_{22} = \{0\}$, we have $H_{22} = H_1 \oplus (H_2 \cap H_{22})$. Therefore: $H^n = H_{11} \oplus H_{22} = H_{11} \oplus H_1 \oplus (H_2 \cap H_{22}) = (H_1 \oplus H_{11}) \oplus (H_2 \cap H_{22})$:

When $H_{11} \subset H_{22}$, the same result follows by using $H_2 = H_{11} \oplus (H_2 \cap H_{22})$

We now give some examples to illustrate the construction of matrices of orthogonal projections.

Example 3.0.2. Find the matrix for the orthogonal projection $P : \mathbb{R}^3 \rightarrow W$ given that W is generated by the vectors $v_1 = (1, 1, 1)$ and $v_2 = (1, 0, 1)$. To see this let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A^T =$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } (A^T A)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$\begin{aligned} \text{Therefore, } Q &= A(A^T A)^{-1} A^T. \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \text{ for any point } (x, y, z) \in \mathbb{R}^3. \end{aligned}$$

$$Q(x, y, z) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{x+z}{2}, y, \frac{x+z}{2} \right).$$

Remark 3.0.3.

Suppose that H^n is an n -dimensional Hilbert space, then $P(H^n)$ and $P_0(H^n)$ shall be used to denote the set of all projections acting on H^n and the set of all orthogonal projections acting on H^n respectively. Naturally, $P_0(H) \subset P(H) \subset B(H)$. The $B(H)$ used in this study is commutative. We begin by discussion of the properties of $P(H^n)$ and $P_0(H^n)$.

Theorem 3.0.4. Suppose that $P, Q \in P_0(H)$ onto H_1 and H_{11} respectively, then the following are equivalent:

- (i). $P - Q$ is an orthogonal projection onto $H_{11} \cap H_1^\perp$.
- (ii). $PQ = QP = P$.
- (iii). $H_{22} \subset H_2$.

Proof.

(i). \Rightarrow (ii). Suppose that $P, Q \in P_0(H)$.

By the projection property, $(Q - P)^2 = Q - P \Rightarrow 2P = PQ + QP$.

Now, $Q(2P = PQ + QP) \Rightarrow 2QP = QPQ + Q^2P$ and

$(2P = PQ + QP)Q \Rightarrow 2PQ = PQ2 + QPQ$ which means that $PQ = QP = P$.

(ii). \Rightarrow (iii). For any $x \in H^n$, $Px \in H_1 \Rightarrow Px = QPx \in H_{11}$ which means that $H_1 \subset H_{11}$.

Suppose $T_m = I_n - P_j$ ($j = 1, 2, \dots$) then $PQ = P$ where $T_1 = I_1 - P$, $T_2 = I_2 - Q$ and $T_1T_2 = T_2$ therefore $T_2x \in H_{22} \Rightarrow T_2x = T_1T_2x \in H_2$ so that $H_{22} \subset H_2$.

(iii). \Rightarrow (ii). Given that $H_{22} \subset H_2$, then for every $x \in H_n$, $Px \in H_1 \subset H_{11}$ which implies that $Q(Px) \Rightarrow QP = P$ and since $H_{22} \subset H_2$, then $T_2x \in H_{22} \subset H_2$ for $x \in H^n \Rightarrow T_1T_2x = T_2x = T_1T_2T_2 \Rightarrow (I_n - P)(I_n - Q) = (I_n - Q) \Rightarrow PQ = P$

(ii). \Rightarrow (i). For $x \in (H_{11} \cap H_2)$, then $(Q - P)x = T_1Qx = T_1x = x$.

But suppose that $x = x_1 + x_2$ where $x_1 \in H_1$ and $x_2 \in H_{22}$ then

$$\begin{aligned} (Q - P)x &= (Q - P)x_1 + (Q - P)x_2 \\ &= QT_1x_1 + T_1Qx_2 = 0. \end{aligned}$$

Therefore $(Q - P)$ is an orthogonal projection onto $H_{11} \cap H_2$ along $H_1 \oplus H_{22}$. ($H_2 = H_1^\perp$ and $H_{22} = H_{11}^\perp$). Now taking $X \in B(H)$, for a commutative $B(H)$, $X(P - Q) = XP - XQ = PX - XQ$ which is the desired derivation. Suppose that $p_n = \{f_i\}_{i=1}^k$ and $q_n = \{g_j\}_{j=1}^k$ are bases for H_1 and H_2 respectively with $H^n = H_1 \oplus H_2$ and $P : H^n \rightarrow H_1, Q : H^n \rightarrow H_2$ then, $\{f_i\}_{i=1}^k - \{g_j\}_{j=1}^k = \gamma$ is a basis for $H_{11} \cap H_1^\perp$ which is the range for $P - Q$.

Corollary 3.0.5.

Let $P, Q \in P_0(H)$, then the operator $PX - XQ$ gives the shortest distance between $H_{11} \cap H_1^\perp$ and H^n .

Proof.

First we recall that $PX - XQ$ projects every point in H^n orthogonally to $H_{11} \cap H_1^\perp$. Let $x_1, x_2 \in (H_{11} \cap H_1^\perp) \subset H^n$, therefore for arbitrary $y \in H^n$, then $\|y - x_1\|^2, \|y - x_2\|^2 <$

$\text{dist}(y, H_{11} \cap H_1^\perp)^2 + \varepsilon$. Recall that $(H_{11} \cap H_1^\perp)$ and $\text{dist}(H_{11} \cap H_1, H) = \inf_{x' \in H_{11} \cap H_1^\perp} \|$

$y - x'\|$. So, by application of parallelogram law,

$$\begin{aligned} \|x_1 - x_2\|^2 &= 2(\|y - x_1\|^2 + \|y - x_2\|^2) - 2\|y - \frac{1}{2}(x_1 + x_2)\|^2 \\ &\leq 2\varepsilon. \end{aligned}$$

Therefore, there exists $t \in H_{11} \cap H_1^\perp$ $t = (P - Q)x$ for some $x \in H^n$ such that $\|y - t\| = \text{dist}(y, H_{11} \cap H_1^\perp)$. So, the approximant of H^n to $H_{11} \cap H_1^\perp$ is the orthogonal projection $P - Q$.

Therefore, every $y \in H$ can be uniquely written as $y = x + x'$ where $x' \in (H_{11} \cap H_1^\perp)$ and $x \in (H_{11} \cap H_1^\perp)$.

Lemma 3.0.6.

Given a compact operator $X \in B(H)$, then PX and XQ are also compact for $P, Q \in P_0(H)$

Proof. Suppose that $X \in B(H)$ is compact and $Q \in P_0(H^n)$ then P is bounded. Let $x_n \in H^n$ be a bounded sequence. Then XQ is also bounded and contains a convergent subsequence. So XQ is compact. Now since X is compact, therefore Xx_n contains a convergent subsequence Xx_{n_k} which converges in the range of X . So PXx_{n_k} also converges.

Theorem 3.0.7. Suppose that $P, Q \in P_0(H^n)$ and a compact $X \in B(H)$, then $\delta_{P,Q}(X)$ is compact.

Proof.

Let there exist bases p_n, q_n and b_n in H^n for P, Q and X respectively in H^n so that $(PX - XQ)$ takes the form, $\gamma = p_n b_n - b_n q_n$. Let γ be compact, U a closed unit ball of $(H_{11} \cap H_1^\perp)$ and z_n a sequence of $\gamma(U)$. It suffices to show that there exists a subsequence of x_n that converges to U .

By the supposition that γ is compact, for every $n \in \mathbb{N}$, $z_n = \gamma x_n$ and x_n belongs to the set U . So, there exists a subsequence x_{n_k} which converges weakly to $x \in U$. We show that γx_{n_k} converges to γx_n . Let γ_n be a sequence of finite rank operator that converges to γ . For any $m' \in \mathbb{N}$, $\gamma_{m'}$ is a closed set which is bounded in a finite dimensional subspace $(H_{11} \cap H_1^\perp)$ of H^n , hence compact. So $\gamma_{m'} x_{n_k}, k \in \mathbb{N}$, converges to $\gamma_{m'}(x)$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\gamma - \gamma_N\| < \frac{\varepsilon}{3}$. Furthermore, given a fixed N , then $k' \in \mathbb{N}$, so that $\|\gamma_N x_{n_k} - \gamma_N x\| \leq \frac{\varepsilon}{3}$ for $k \geq k'$. So that

$$\|\gamma_N x_{n_k} - \gamma x\| \leq \|(\gamma - \gamma_N)x_{n_k}\| + \|\gamma_N(x_{n_k} - x)\| + \|(\gamma_N - \gamma)x\|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

So $z_{n_k} = \gamma x_{n_k}$ converges to $\gamma x \in \gamma(U)$ so that $\gamma(U)$ is compact. Suppose that $\gamma(U)$ is compact, then the union $\cup_{z_n \in \gamma(U)} B(z_n, \frac{1}{n})$ is an open covering of the compact set $\gamma(U)$, and therefore we can obtain vectors $x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)} \in H^n$ such that $\cup_{i=1}^k U(\gamma x_i^{(n)}, \frac{1}{n})$ is a covering of $\gamma(U)$. Suppose that H^n span $\gamma x_i^{(n)}, i \in \mathbb{N}, T$ an arbitrary orthogonal projection on H^n . Let also $\gamma_n = T_n \gamma$.

For $\epsilon > 0$, and $N > \frac{1}{\epsilon}$ if $n \geq N$ with $\|x\| \leq 1$, then $\|\gamma x - \gamma_n x\| = \|\gamma x - T \gamma x\|$.

Now because $T \gamma \in \gamma_n$ is the point in H^n closest to γx , therefore $\|\gamma x - \gamma_n x\| \leq \inf_{1 \leq i \leq n} \|\gamma x - \gamma x_i^{(n)}\| < \frac{1}{n} < \epsilon$.

So, $\gamma_n \rightarrow \gamma$ as $n \mapsto \infty$. Which implies that $D_{op}^G [B(H)]$ is a compact ideal of $B(H)^n$. Now since $\gamma_n \rightarrow \gamma \in D_{op}^G [B(H)]$ the assertion is proved.

Example 3.0.8. Let $H^n = \ell^2$ and $m \times m$ operators $P = [a_{ij}]$ and $Q = [b_{ij}]$ such that

$$a_{ij} = \begin{cases} p_n, & i = j \\ 0, & i \neq j \end{cases} \text{ and } b_{ij} = \begin{cases} q_n, & i = j \\ 0, & i \neq j \end{cases}$$

for $n = m - 1, q_n^* (q_n - 1) = 0$ and $m \geq 2$.

Let the operators P and Q be bounded i.e. for $n \geq 1, p_n = (p_1, p_2, \dots) \in \ell^\infty$ and $q_n = (q_1, q_2, \dots) \in \ell^\infty$. Let P be majorized by Q or Q majorized by P so that $(p_n - q_n)$ is also diagonal and $(p_n - q_n) = ((p_1 - q_1), (p_2 - q_2), \dots) \in \ell^\infty$. Suppose that $\lim_{n \rightarrow \infty} (p_n - q_n) = 0$ and $(P - Q)_n = \text{diag}((p_1 - q_1), (p_2 - q_2), 0, 0, \dots)$, then $(P - Q)_n$ is compact and $\|(P - Q) - (P - Q)_n\| = \sup\{|p_n - q_n| \geq n + 1\} \rightarrow 0$. For an arbitrary $X \in B(\ell^2)^2$, then $PX - XQ$ is also compact. Suppose that $x_n \in \ell^2$, with the following conditions, $\|x_n\| \leq 1, \|(PX - XQ)x\| = \|PX - XQ\|$ and some $\alpha, \beta \in \mathbb{F}$, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle (p_n - q_n)x_n, x_n \rangle &= \lim_{n \rightarrow \infty} \langle (p_n - q_n)^2 x_n, x_n \rangle \\ &= \lim_{n \rightarrow \infty} \langle (p_n - q_n)^* x_n, x_n \rangle \\ &= |\alpha - \beta|. \end{aligned}$$

Thus $|\alpha - \beta| \in \mathbb{R}^+$.

Example 3.0.9. Let $H^n = L^2(\mathbb{T})$ be the space of 2π -periodic functions and a constant function $u = \frac{1}{\sqrt{2\pi}}$, with $\|u\| = 1$, then the orthogonal projections Pu and Qu are defined by $Puf = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$ and $Quf = \frac{1}{2\pi} \int_0^{2\pi} g(x) dx$. So $P - Q = \frac{1}{2\pi} \int (f(x) - g(x)) dx$ and so for $X \in B(L^2(\mathbb{T}))$, then $PX - XQ = \frac{1}{2\pi} \int_0^{2\pi} b(x)(f(x) - g(x)) dx$ is compact.

We apply the following example in showing how a matrix of $\delta_{P,Q}$ can be constructed.

Example 3.1.0. Consider two sets of vector $v_1 = (0, 1, 0)$, $v_2 = (0, 1, 1)$

and $u_1 = (1, 1, 1)$, $u_2 = (1, 0, 1)$. By simple calculation, we get that

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, A^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, (AA^T)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

so $A(A^T A)^{-1} A^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ thus we get an orthogonal projection

$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Similarly for the second set of vectors, we get another orthogonal projection

$XQ = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$. Now for an arbitrary operator with a matrix representation

$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ then $PX = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $XQ = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that

$$PX - XQ = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{-1}{2} \\ \frac{-1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 3.1.1. Let H^n be a complex four-dimensional Hilbert space and $B(H)$ algebra of 4×4 matrices. We take $P_0(H)$ to be the subalgebra of diagonal matrices, so $\delta_{P,Q} : P_0(H) \rightarrow B(H)$.

Suppose that $P, Q \in P_0(H)$ are selfadjoint orthogonal projections onto H_1 and H_{11} spanned by the orthogonal unit vectors

$$x_1 = \frac{1}{2\sqrt{3}} (1, -1 + i\sqrt{3}, \frac{1}{\sqrt{14}}(4 - 5i\sqrt{3}), \frac{1}{\sqrt{14}}(-2 + i\sqrt{3}))$$

$$x_2 = \frac{1}{2\sqrt{3}} (1, -1 - i\sqrt{3}, \frac{1}{\sqrt{14}}(-2 - i\sqrt{3}), \frac{1}{\sqrt{14}}(4 + 5i\sqrt{3}))$$

And the unit vector

$$x_3 = \frac{1}{2\sqrt{3}} (1, 2, \sqrt{\frac{7}{2}}, \sqrt{\frac{7}{2}})$$

respectively.

Then for an arbitrary operator $X \in B(H)$ with $\|X\| = 1$, operator $PX - XQ$ has a Hermitian matrix

Given by

$$\frac{1}{12} \begin{bmatrix} 1 & -4 & \frac{1}{\sqrt{14}}(-5 + 6i\sqrt{3}) & \frac{1}{\sqrt{14}}(-5 - 6i\sqrt{3}) \\ -4 & 4 & -2\sqrt{14} & -2\sqrt{14} \\ \frac{1}{\sqrt{14}}(-5 - 6i\sqrt{3}) & -2\sqrt{14} & \frac{7}{2} & \frac{1}{14}(-95 + 12i\sqrt{3}) \\ \frac{1}{\sqrt{14}}(-5 + 6i\sqrt{3}) & -2\sqrt{14} & \frac{1}{14}(-95 - 12i\sqrt{3}) & \frac{7}{2} \end{bmatrix}$$

which is also idempotent. We now consider the linearity of $\delta_{P,Q}$ in the following proposition.

Proposition 3.1.2. A derivation $PX - XQ$ is linear for an arbitrary $X \in B(H)$.

Proof.

Let $\{f_i\}_{i \in I}$ and $\{q_i\}_{i \in J}$ be two orthonormal bases for $H_{11} \cap H_1^\perp$ and $(H_{11} \cap H_1^\perp)^\perp$, respectively, and $T = I - (P - Q)$ be the orthogonal projection on $(H_{11} \cap H_1^\perp)^\perp$. Suppose $X \in B(H)$ then for $x_1, x_2 \in H$ and $\alpha, \beta \in \mathbb{K}$, then by theorem 4.12, there exist $\gamma = (p_n b_n - b_n q_n)$ such that

$$\begin{aligned} \gamma(\alpha x_1 + \beta x_2) &= \gamma \sum_{i \in I} \sum_n \langle \alpha x_1 + \beta x_2, f_i \rangle \\ &= \sum_{i \in I} \sum_n \langle (p_n b_n - b_n q_n)(\alpha x_1 + \beta x_2), f_i \rangle \\ &= \sum_{i \in I} \sum_n \langle \alpha p_n b_n x_1 + \beta p_n b_n x_2 - \alpha b_n q_n x_1 - \beta b_n q_n x_2, f_i \rangle \\ &= \alpha \sum_{i \in I} \sum_n \langle p_n b_n x_1, f_i \rangle + \beta \sum_{i \in I} \sum_n \langle p_n b_n x_2, f_i \rangle - \alpha \sum_{i \in I} \sum_n \langle b_n q_n x_1, f_i \rangle - \beta \sum_{i \in I} \sum_n \langle b_n q_n x_2, f_i \rangle \\ &= \alpha \sum_{i \in I} \sum_n \langle (p_n b_n - b_n q_n)x_1, f_i \rangle + \beta \sum_{i \in I} \sum_n \langle (p_n b_n - b_n q_n)x_2, f_i \rangle \\ &= \alpha \gamma x_1 + \beta \gamma x_2 \end{aligned}$$

Corollary 3.1.3. A derivation $PX - XQ$ is linear on (i) H^n and (ii) $B(H^n)$.

Proof.

(i). Linearity in H^n : Given that $P, Q \in P_0(H)$, by [76, lemma 2] we can obtain a pair $x, y \in H$ and $\alpha, \beta \in \mathbb{K}$ and on setting $\lim_n \|Px_n\| \Rightarrow \|P\|$, $\lim_n \|Qx_n\| \Rightarrow \|Q\|$ and $\lim_n \langle Qx_n, x_n \rangle \rightarrow |\mu|$, $\lim_n \langle Px_n, x_n \rangle \rightarrow |\lambda|$ and also setting $Qx = \alpha x + \beta y$, $Px = \alpha^* x + \beta^* y$ with $\langle x, y \rangle = 0$ and $\|x\| = \|y\| = 1$. Set also that $Xx = x$, $Xy = -y$ and also that X acts on $\{x, y\}$ then, $P - Q$ is an orthogonal projection onto $H_{11} \cap H_1^\perp$. On respective post and premultiplication of P and Q of $P - Q$ by X gives a new operator of the form γ such that for $x \in H$,

$$\begin{aligned} \gamma x &= Px - \alpha^* Xx + \beta^* Xy \\ &= \alpha x + \beta y - \alpha^* x + \beta^* y \\ &= (\alpha - \alpha^*)x + (\beta + \beta^*)y, \text{ for } \alpha x, \beta y \in H^n \end{aligned}$$

and on the other hand

$$\delta_{P,Q}(X)(\alpha x + \beta y) = \gamma(\alpha x + \beta y)$$

$$\begin{aligned}
 &= \gamma\alpha x + \gamma\beta y \\
 &= \alpha\delta_{P,Q}(X)x + \beta\delta_{P,Q}(X)y
 \end{aligned}$$

(ii). Linearity in $B(H^n)$: By similar calculation, we can find a pair of operators $X, Y \in B(H)$ and $\alpha, \beta \in \mathbb{K}$

$$\begin{aligned}
 \delta_{P,Q}(\alpha X + \beta Y)x &= \{P(\alpha X + \beta Y) - (\alpha X + \beta Y)Q\}x \\
 &= \{\alpha PX + \beta PY - \alpha XQ - \beta YQ\}x \\
 &= \alpha PX - \alpha XQ + \beta PY - \beta YQx \\
 &= \{\alpha\delta_{P,Q}(X)x + \beta\delta_{P,Q}(Y)\}x.
 \end{aligned}$$

So, $\delta_{P,Q}X$ is linear on both H^n and $B(H)$.

Proposition 3.1.4. Suppose that $P, Q \in P_0(H)$ and an arbitrary $X \in B(H)$, then $\delta^2_{P,Q}(X) = PX + XQ - 2PXQ$.

Proof.

$$\begin{aligned}
 \delta^2_{P,Q}(X) &= \delta_{P,Q}(PX - XQ) \\
 &= P(PX - XQ) - (PX - XQ)Q \\
 &= PPX - PXQ - PXQ + XQQ \\
 &= PPX + XQQ - 2PXQ \\
 &= PX + XQ - 2PXQ
 \end{aligned}$$

Theorem 3.1.5. Suppose that $P, Q \in P_0(H)$, then the derivation $\delta_{P,Q}(X) = PX - XQ$ is bounded from below.

Proof.

By the definition of $\delta_{P,Q}$ we observe that $\delta_{P,Q}(X) = p_n - q_n$ is meaningful for the bases p_n and q_n of P and Q respectively with $\sum_n \|f_n\|^2 = 1$. Since p_n and q_n are bounded, from the definition of $\delta_{P,Q}(X)$, we have,

$$\begin{aligned}
 \|\delta_{P,Q}(f_n)\|^2 &= \|\sum_n (p_n f_n - f_n q_n)\|^2 \\
 &\geq \sum_n \|p_n f_n\|^2 - \sum_n \|f_n q_n\|^2 \\
 &= \{\sum_n |p_n|^2 - \sum_n |q_n|^2\} \|f_n\|^2.
 \end{aligned}$$

Since the difference of finite summation of p_n and q_n is also bounded, by taking supremum of both sides of the above inequality gives $\|\delta\| \geq \{\sum_n |p_n|^2\}^{1/2} - \{\sum_n |q_n|^2\}^{1/2}$.

Proposition 3.1.6. Suppose that $P, Q \in P_0(H)$, then the derivation $\delta_{P,Q}$ is bounded from above.

Proof. Let P, Q and X be induced by p_n, q_n respectively and f_n as arbitrary elements of $B(H)$.

By the definition of δ , we have for $\{\|\sum_n f_n\|^2\} = 1$ and $\|\sum_n f_n\| \leq 1$ that

$$\|\delta_{P,Q}\|^2 = \|\sum_n (p_n f_n - f_n q_n)\|^2$$

$$\begin{aligned} &\leq \| \sum_n p_n f_n \|^2 + \| \sum_n f_n q_n \|^2 \\ &\leq \{ \sum_n |p_n|^2 + \sum_n |q_n|^2 \} \{ \| \sum_n f_n \|^2 \} \\ &= \{ \sum_n |p_n|^2 + \sum_n |q_n|^2 \} \\ &= \{ \sum_n |p_n| + \sum_n |q_n| \}. \end{aligned}$$

Taking the supremum of both sides of the inequality gives us $\| \delta_{P,Q}(X) \| \leq \{ \sum_n |p_n| \}^{\frac{1}{2}} + \{ \sum_n |q_n| \}^{\frac{1}{2}}$.

Proposition 3.1.7. Suppose that $P, Q \in P_0(H)$, then $\delta_{P,Q}(X)$ has a bounded inverse on $H_{11} \cap H_1^\perp$ if and only if $\delta_{P,Q}(X)$ is bounded from below.

Proof.

From

the definition of $\delta_{P,Q}$, we have that $\delta_{P,Q}(X)$ is a transformation $\delta_{P,Q}(X): H^n \rightarrow H_{11} \cap H_1^\perp$. Now suppose that p_n, q_n and x_n as described in theorem 4.12 are all bounded from, then so is γ , and therefore there exists a real number $m > 0$ such that $\| \gamma(x) \| \geq m \| x \| \forall x \in H^n$. This means that γ is a one-to-one map. Thus γ is a bijection and hence has an inverse, $\gamma^{-1} : B(H) \rightarrow P_0(H)^n$ which is linear and onto. We then show that γ^{-1} is bounded and $\| \gamma^{-1} \| \leq \frac{1}{m}$.

Let $y \in H_{11} \cap H_1^\perp$ and $P', Q' \in P_0(H)^n$, then $y \in \gamma(x)$, for unique elements $x \in H^n$ and $P, Q \in P_0(H)^n$. Now, since γ is bounded from below, we get $m^{-1} \| y \| \geq \| \gamma^{-1}y \|$ i.e $\| \gamma^{-1}y \| \leq m^{-1} \| y \|$ and since P, Q are arbitrary in $P_0(H)^n$ and y is arbitrary in $H_{11} \cap H_1^\perp$, we get $\| \gamma^{-1}y \| \leq \frac{1}{m} \| y \| \forall y \in H_{11} \cap H_1^\perp$. Thus γ^{-1} is bounded. Also $\| \gamma^{-1} \| \leq \frac{1}{m}$.

Conversely, suppose that γ has a bounded inverse on $P_0(H)^n$. Since $H^n \neq 0$, we have $\| \gamma^{-1} \| \neq 0$ and therefore $\| \gamma^{-1} \| > 0$. Since $\gamma : P_0(H)^n \rightarrow B(H)^n$ is bijective, each $y \in H_{11} \cap H_1^\perp$ is $\gamma(x)$ for a unique $x \in H^n$. So the relation $\| \gamma^{-1} \| \leq \| \gamma^{-1} \| \| y \| \forall y \in H_{11} \cap H_1^\perp$ can be written as $\| \gamma^{-1} \gamma(x) \| \geq \| \gamma^{-1} \| \| x \| \forall y \in H_{11} \cap H_1^\perp$.

Which shows that $\delta_{P,Q}(X)$ is bounded from below.

Corollary 3.1.8. Given $P, Q \in P_0(H)$ then $\delta_{P,Q}$ is continuous.

Proof.

First we assume that $P \perp Q$. For an arbitrary $x \in H^n, \| x \| = 1$, then $x = Px + Qx$ and $\| x \|^2 = \| Px \|^2 + \| Qx \|^2 \geq \| Px \|^2$ and $\| x \|^2 = \| Px \|^2 + \| Qx \|^2 \geq \| Qx \|^2$. Thus both Px and Qx are bounded by 1 and so is PX and QX . Suppose that $\delta_{P,Q}$ is continuous at 0, then we can get some $\lambda > 0$ such that for all $y \in H$ with $\| y \| < \lambda$ then $\| \gamma y \| < 1$. Now for $x \in H$ and $x \neq 0$ then $\lambda \left(\frac{x}{2\|x\|} \right) = \frac{\lambda}{2}$, so $\| \gamma \left(\lambda \frac{x}{2\|x\|} \right) \| < 1$. By the linearity of $PX - XQ$ and homogeneity of the norm,

we get

$$1 \geq \| \gamma \left(\lambda \frac{x}{2\|x\|} \right) \| = \| \lambda \frac{\gamma x}{2\|x\|} \| = \frac{\lambda}{2\|x\|} \| \gamma x \|$$

and therefore $\| (PX - XQ)x \| \leq M \| x \|$ with $M = \frac{2}{\lambda}$.

In the following discussion, we consider the positivity of the operator $\delta_{P,Q}$ on H^n .

Lemma 3.1.9. The product of two commuting positive operators $P \in P_0(H^n)$ and $X \in B(H)$ on H^n is also positive on H^n .

Proof.

Let $P \neq 0$ and define a sequence of operators $\{S_{n=1}^\infty\}$ by $S_1 = P \| P \|$, $S_{n+1} = S_n - S_n^2 = S_n(I - S_n)$ for polynomials S_n in P and hence selfadjoint operators that commute with P for all $n \in N$. So that $P = \| P \| \sum_{n=1}^\infty S_n^2$. For every $x \in H^n$, $\sum_{n=1}^\infty \| S_n x \|^2 < \infty$ so that $\| S_n x \| \rightarrow 0$. Now, $\langle PXx, x \rangle = \| P \| \sum_{n=1}^\infty \langle XS_n x, S_n x \rangle \geq 0$.

Lemma 3.2.0. Let H^n be a finite dimensional Hilbert space and $P, Q \in P_0(H^n)$ such that $ranP \subseteq ranQ$ and $X \in B(H)$ a positive operator on H^n that commutes with both P and Q .

Then $\| PX - XQ \|^2 \geq \| PX \|^2 - \| XQ \|^2$ and $\| PX \|^2 \leq \| XQ \|^2$

Proof.

We invoke vector majorization thus: Given that $P, Q \in P_0(H)$ then $P - Q$ is an orthogonal projection onto $H_{11} \cap H_1^\perp$ along $(H_{11} \cap H_1^\perp)^\perp$ and $ranP, ranQ \in H_{11} \cap H_1^\perp$. Let $p \in ranP$, $p = \{p_i\}_{i=1}^m$ and $q \in ranQ$, $q = \{q_i\}_{i=1}^n$. For suitable bases, we can obtain the matrices for P, Q and $X \in B(H)$ such that the Hilbert-Schmidt norm applies as follows; $\| P \|_2 = (\sum_{i=1}^m \sum_{j=1}^n |p_{ij}|^2)^{\frac{1}{2}}$, $\| Q \|_2 = (\sum_{i=1}^m \sum_{j=1}^n |q_{ij}|^2)^{\frac{1}{2}}$ and $\| X \|_2 = (\sum_{i=1}^m \sum_{j=1}^n |x_{ij}|^2)^{\frac{1}{2}}$. Then $\sum_i^k q[i] \leq \sum_i^k p[i]$ and for an arbitrary $\sum_i^k x[i]$.

$$\begin{aligned} \| PX - XQ \|_2 &= (\sum_{i=1}^m \sum_{j=1}^n |p_{ij}x_{ij} - x_{ij}q_{ij}|^2)^{\frac{1}{2}} \\ &\geq (\sum_{i=1}^m \sum_{j=1}^n |p_{ij}x_{ij}|^2)^{\frac{1}{2}} - (\sum_{i=1}^m \sum_{j=1}^n |q_{ij}x_{ij}|^2)^{\frac{1}{2}} \\ &= (\sum_{i=1}^m \sum_{j=1}^n |p_{ij}x_{ij}|^2)^{\frac{1}{2}} - (\sum_{i=1}^m \sum_{j=1}^n |x_{ij}q_{ij}|^2)^{\frac{1}{2}} \\ &= \| PX \|_2 - \| XQ \|_2 \leq 0 \end{aligned}$$

Theorem 3.2.1. Let $P, Q \in P_0(H)$ such that $P \geq Q$ and an arbitrary positive operator $X \in B(H)$. Then $\delta_{P,Q}(X) = PX - XQ$ is positive.

Proof.

It suffices to show that $\delta_{P,Q}(X)$ has square roots. If $P = Q$ then $\delta_{P,Q}(X) = 0$ then $\delta_{P,Q}(X) \geq 0$, non-negative. Suppose that $P - Q \neq 0$ then $0 \leq (P - Q) \leq I \Rightarrow 0 \leq (PX - XQ) \leq I$ for an arbitrary positive operator $X \in B(H)$.

Now $\| (PX - XQ) \| - (PX - XQ)$ must then satisfy the condition that $0 \leq \| (PX - XQ) \| - (PX - XQ) \leq I$ and so we can find an operator \tilde{A} such that $\tilde{A}^2 = \| (PX - XQ) \| - (PX - XQ)$ then $S = (\sqrt{\| (PX - XQ) \|})\tilde{A}$ satisfies $S^2 = (PX - XQ)$. We then set $Z = I - (PX - XQ)$

and $V = I - S$. The operator V should have the property $(I - V)^2 = I - Z$, that is implicitly expressed as

$$V = \frac{1}{2}(Z + V^2) \dots\dots\dots(4.2.1)$$

Now $0 \leq Z \leq I$, and V chosen is such that $0 \leq V \leq I$.

Conversely, if $0 \leq V \leq I$ and satisfy equation (4.2.1) above, then $S = I - V$ is a positive square root of T . We apply method of successive approximations to solve (4.2.1). We set $V_0 = I$ and define V_n recursively by

$$V_{n+1} = \frac{1}{2}(Z + V_n^2), \quad n = 0, 1, 2, \dots\dots\dots (4.2.2).$$

We show that V_n converges strongly to a solution of equation (4.2.1).

$$\text{Let } 0 \leq V_n \leq I \dots\dots\dots(4.2.3)$$

This is obviously true for a positive integer n .

$$\langle V_{n+1} x, x \rangle = \frac{1}{2} \langle Zx, x \rangle + \frac{1}{2} \| V_n x \|^2 \quad \forall x \in H^n, \dots\dots\dots(4.2.4)$$

which implies that $V_{n+1} \geq 0$

Now $V_0 < 1$. Suppose that $V_n \leq I$, then equation (4.2.4) gives

$$\langle V_{n+1} x, x \rangle \leq \frac{1}{2} \langle Ix, x \rangle + \frac{1}{2} \| x \|^2 = \langle Ix, x \rangle \quad (\text{since } Z \leq I) \text{ and } V_n \leq I. \text{ Thus } V_{n+1} \leq I.$$

Consequently, $\| V_n \| \leq I \quad \forall n \in \mathbb{N}$. Now we show that $V_n \leq V_{n+1} \quad \forall n \in \mathbb{N}, V \setminus \{0\}$ i.e $V_{n+1} - V_n \geq 0$. Next, we observe that V_n is a polynomial in Z with non-negative coefficients. Now this is true for $n = 0$ (for $V_1 - V_0 = \frac{1}{2}(Z - I)$). We observe that

$$V_{n+1} - V_n = \frac{1}{2}(Z + V_n^2) - \frac{1}{2}(Z + V_{n-1}^2) \dots\dots\dots(4.2.5)$$

(It is noted that V_{n-1} and V_n are both polynomials in Z and so $V_n \leftrightarrow V_{n-1}$). Suppose that $V_n - V_{n-1}$ is a polynomial in Z with non-negative coefficients, then equation (4.2.5) shows that $V_{n+1} - V_n$ is also a polynomial in Z with non-negative coefficients for each non-negative n .

Next, we show that

$$Z^k \geq 0. \dots\dots\dots(4.2.6)$$

For $k = 0, 1, 2, \dots$ If $k = 2j$, then $\langle Z^k x, x \rangle = \| Z^j x \|^2 \geq 0, \quad \forall x \in H^n$. Using equation (4.2.6) and the fact that each $V_{n+1} - V_n$ is a polynomial in Z with non-negative coefficients, we see that $V_{n+1} - V_n \geq 0$ for all the non-negative integer n . The sequence (V_n) satisfies,

$$0 \leq V_n \leq V_{n+1} \leq I, \quad n = 0, 1, 2, \dots\dots\dots(4.2.7)$$

and so there is a self-adjoint operator $V \in B(H)$ such that

$$V_n \leftrightarrow V, \quad V_n \leq V \leq I, \quad n = 0, 1, 2, \dots\dots\dots(4.2.8)$$

By equation (4.2.8), we see that the operator V is a solution of equation (4.2.4). Letting $n \rightarrow \infty$

we have from equation (4.2.2)

$$\begin{aligned}
 V &= S - \lim V_{n+1} \\
 &= S - \lim \frac{1}{2}(Z + V_n^2) \\
 &= \frac{1}{2}(Z + V_n)
 \end{aligned}$$

then $S = I - V$ is a square root of $(PX - XQ)$.

3. Conclusion

We have shown that $PX - XQ$ is bounded, continuous everywhere and positive i.e. $\|PX - XQ\| \geq 0$ for positive operators P, Q and an arbitrary operator X . For objective two, we have approximated the norm of $\delta_{P,Q}$ by the formula $\|\delta_{P,Q}\| = \{\sum |\alpha|^2\}^{\frac{1}{2}} - \{\sum |\beta|^2\}^{\frac{1}{2}}$ and that this norm is bounded.

References

- [1] Anderson J. H. and Foias C., Properties which normal operators share with normal derivation and related operators, Pacific J. Math., 61 (1976), 313-325.
- [2] Cabello J. C. and Peralta A. M., Weak-2-local symmetric maps on C^* -algebras, Linear Algebra Appl. 494(2016), 32-43.
- [3] Jorda E. and Peralta A. M., Stability of derivations under weak 2-local continuous perturbations, Aequationes Mathematicae, 2016.
- [4] Curto R. E., The Spectra of elementary operators, Indiana University Mathematics Journal, Indiana, (1983).
- [5] Johnson B. E., Continuity of centralizers on Banach algebras, Amer. J. Math., 91 (1969), 1-10
- [6] Kadison R. V., Lance E. C. and Ringrose J. R. Derivations and Automorphisms of operator algebras II, J. of Functional Analysis 1, (1947), 204-221.
- [7] Kadison R. V. Local derivations, J. d'lgebra., 130(1990), 494-509.
- [8] Kaplansky I. M., Modules over operators algebras, Amer. J. Math., 75 (1953), 839-858.
- [9] Matej B. Characterizations of Derivations on Some Normed Algebras with Involution, Journal Of Algebra, 152 (1992), 454-462.
- [10] Niazi M. and Peralta A. M., Weak-2-local derivations on M_n , Jstor, 31(6) (2017), 1687-1708.
- [11] Ptak V., Derivations, commutators and radicals, Manuscripta. Math., 23 (1978), 355-362.
- [12] Ringrose J. R., Automatic continuity of derivations of operator algebras J. London Math. Soc., 5(2) (1972), 432.

- [13] Salah M., Generalized derivations and C^* -algebras An. St. Univ. Ovidius Constanta, 17(2) (2009), 123-130.
- [14] Salah M., Some recent results on operator commutators and related operators with applications.
- [15] Sakai, Shoichiro, On conjecture of Kaplansky Tohoku Math J., 12(2) (1960), 31-33.
- [16] Semrl P., Local automorphisms and derivations on $B(H)$, Proc. Amer. Math Soc. 125 (1997) 2677-2680.
- [17] Sinclair A.M., Continuity of derivations and a problem of Kaplansky, Amer. J. Math., 90 (1968), 1067-1073.
- [18] Sommerset D.W.B., The proximality of the centre of a C^* -algebra, J. Approx. Theory, 89 (1997), 114, 117.
- [19] Yanai H., Takeuchi K. and Takane Y., Projection matrices, generalized inverse matrices and singular value decomposition, Springer, New York, 2011.