Existence of Solutions for a Nonhomogeneous Navier Problems With Degenerated Operators

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Abstract

In this paper we are interested in the existence of solutions for Navier problem associated with the degenerate nonlinear elliptic equations

$$\begin{cases} \Delta\big[\omega_1\,|\Delta u|^{p-2}\Delta u + \omega_2|\Delta u|^{q-2}\Delta u\big] \\ -\operatorname{div}\big[\mathcal{A}(x,u,\nabla u)\,\omega_3 + \mathcal{B}(x,u,\nabla u)\,\omega_4\big] \\ +\mathcal{H}(x,u,\nabla u)\,\omega_5 = f_0(x) - \sum_{j=1}^{n} D_j f_j(x) \ \text{ in } \ \Omega, \\ u(x) = \Delta u(x) = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

in the setting of the weighted Sobolev spaces.

Keywords: degenerate nonlinear elliptic equations, weighted Sobolev spaces, Navier problem.

1 Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $X = W_0^{1,r}(\Omega,\omega_3)\cap W_0^{2,p}(\Omega,\omega_1)$ (see Definition 2.2 and Definition 2.3) for the Navier problem

$$(P)\begin{cases} Lu(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x) \ \text{ in } \ \Omega, \\ u(x) = \Delta u(x) = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where L is the partial differential operator

$$\begin{aligned} Lu(x) &= \Delta\big[\omega_1\,|\Delta u|^{p-2}\Delta u + \omega_2|\Delta u|^{q-2}\Delta u\big] - \operatorname{div}\big[\mathcal{A}(x,u,\nabla u)\,\omega_3 + \mathcal{B}(x,u,\nabla u)\,\omega_4\big] \\ &+ \mathcal{H}(x,u,\nabla u)\,\omega_5, \end{aligned} \tag{1.1}$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2, ω_3, ω_4 and ω_5 are five weight functions (which represent the degeneration or singularity in the equation (1.1)), $2 \le q < p < \infty$, $2 \le s, z < r < \infty$ and the functions $\mathcal{A}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $\mathcal{B}_j : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j = 1, ..., n)$ and $\mathcal{H} : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfying the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$,

$(\eta,\xi)\mapsto\mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$;

(H2) There exists a constant $\theta_1 > 0$ such that

$$\langle\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi'), (\xi-\xi')\rangle \ge \theta_1\,|\xi-\xi'|^r,$$

whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, and $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi), ..., \mathcal{A}_n(x,\eta,\xi))$ (where $\langle.,.\rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^n$);

(H3) $\langle\mathcal{A}(x,\eta,\xi),\xi\rangle \ge \lambda_1|\xi|^r$, where λ_1 is a positive constant;

(H4) $|\mathcal{A}(x,\eta,\xi)| \le K_1(x) + h_1(x)|\eta|^{r/r'} + h_2(x)|\xi|^{r/r'}$, where K_1, h_1 and h_2 are nonnegative functions, with $h_1, h_2\in L^\infty(\Omega)$ and $K_1\in L^{r'}(\Omega,\omega_3)$ (with $1/r + 1/r' = 1$);

(H5) $x\mapsto\mathcal{B}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$,

$(\eta,\xi)\mapsto\mathcal{B}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$;

(H6) There exists a constant $\theta_2 > 0$ such that

$$\langle\mathcal{B}(x,\eta,\xi) - \mathcal{B}(x,\eta',\xi'), (\xi-\xi')\rangle \ge \theta_2\,|\xi-\xi'|^s,$$

whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{B}(x,\eta,\xi) = (\mathcal{B}_1(x,\eta,\xi), ..., \mathcal{B}_n(x,\eta,\xi))$;

(H7) $\langle\mathcal{B}(x,\eta,\xi),\xi\rangle \ge \lambda_2|\xi|^s + \Lambda_2|\eta|^s$, where $\lambda_2 > 0$ and $\Lambda_2 \ge 0$ are constants;

(H8) $|\mathcal{B}(x,\eta,\xi)| \le K_2(x) + g_1(x)|\eta|^{s/s'} + g_2(x)|\xi|^{s/s'}$, where K_2, g_1 and g_2 are nonnegative functions, with g_1 and $g_2\in L^\infty(\Omega)$, and $K_2\in L^{s'}(\Omega,\omega_4)$ (with $1/q + 1/q' = 1$).

(H9) $x\mapsto\mathcal{H}(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$

$(\eta,\xi)\mapsto\mathcal{H}(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$.

(H10) $[\mathcal{H}(x,\eta,\xi) - \mathcal{H}(x,\eta',\xi')](\eta-\eta') > 0$, whenever $\eta,\eta'\in\mathbb{R}$, $\eta\neq\eta'$.

(H11) $\mathcal{H}(x,\eta,\xi)\eta \geq \lambda_3|\xi|^z + \Lambda_3|\eta|^z$, λ_3 and Λ_3 are nonnegative constants.

(H12) $|\mathcal{H}(x,\eta,\xi)| \leq K_3(x) + h_3(x)|\eta|^{z/z'} + h_4(x)|\xi|^{z/z'}$, where K_3, h_3 and h_4 are nonnegative functions, with $K_3 \in L^{z'}(\Omega,\omega_5)$ (with $1/z + 1/z' = 1$), h_3 and $h_4 \in L^\infty(\Omega)$.

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [2], [3], [4] and [7]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaciology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [1] and [6]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [15]). These classes have found many useful applications in harmonic analysis (see [17]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [13]). There are, in fact, many interesting examples of weights (see [12] for p-admissible weights).

The following theorem will be proved in section 3.

Theorem 1.1. *Let $2 \leq q < p < \infty$, $1 < s, z < r < \infty$ and assume (H1)-(H12). If*

(i) $\omega_1 \in A_p$, $\omega_2 \in \mathcal{W}(\Omega)$, and $\dfrac{\omega_2}{\omega_1} \in L^{p/(p-q)}(\Omega,\omega_1)$;

(ii) $\omega_3 \in A_r$, $\omega_4 \in A_s$ and $\dfrac{\omega_4}{\omega_3} \in L^{r/(r-s)}(\Omega,\omega_3)$;

(iii) $\omega_5 \in \mathcal{W}(\Omega)$ and $\dfrac{\omega_5}{\omega_3} \in L^{r/(r-z)}(\Omega,\omega_3)$.

(iv) $f_j/\omega_3 \in L^{r'}(\Omega,\omega_3)$ $(j = 0, 1, ..., n)$.

Then the problem (P) has a unique solution $u \in X = W_0^{1,r}(\Omega,\omega_3) \cap W_0^{2,p}(\Omega,\omega_1)$. Moreover, if

$2 < r < \infty$ we have

$$\|u\|_X \leq \gamma_{p,r}\left(\frac{1}{p'}M^{p'-1} + \frac{1}{r'}\left(\frac{M}{\lambda_1}\right)^{r'-1}\right)$$

where $\gamma_{p,r} = p\,r/(p\,r - p - r)$, $M = C_\Omega \|f_0/\omega_3\|_{L^{r'}(\Omega,\omega_3)} + \sum_{j=1}^{n} \|f_j/\omega_3\|_{L^{r'}(\Omega,\omega_3)}$ and C_Ω is the constant in Theorem 2.2.

2 Definitions and Basic Results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}\,dx\right)^{p-1} \leq C,$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [10], [12] or [17] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x;2r)) \leq C\,\mu(B(x;r)),$$

for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [12]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [17]).

If $\omega \in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C\frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [12]). Therefore, if $\mu(E) = 0$ then $|E| = 0$. The measure μ and the Lebesgue measure $|.|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E) = 0$ if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p \, \omega \, dx \right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [18]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, k be a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p \, \omega \, dx + \sum_{1 \leq |\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, \omega \, dx \right)^{1/p}. \tag{2.1}$$

If $\omega \in A_p$, then $W^{k,p}(\Omega, \omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Corollary 2.1.6 in [18]). We also define the space $W^{k,p}_0(\Omega, \omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.1). We have that the spaces $W^{k,p}(\Omega, \omega)$ and $W^{k,p}_0(\Omega, \omega)$ are Banach spaces.
The space $W^{1,p}_0(\Omega, \omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W^{1,p}_0(\Omega, \omega)$ is a reflexive Banach space (see [14] for more information about the spaces $W^{1,p}(\Omega, \omega)$). The dual of space $W^{1,p}_0(\Omega, \omega)$ is the space

$$[W^{1,p}_0(\Omega, \omega)]^* = \{T = f_0 - \text{div}(F),\ F = (f_1, ..., f_n) : \frac{f_j}{\omega} \in L^{p'}(\Omega, \omega), j = 0, 1, ..., n\}.$$

It is evident that a weight function ω which satisfies $0 < c_1 \leq \omega(x) \leq c_2$ for $x \in \Omega$ (where c_1 and c_2 are constants), give nothing new (the space $W^{1,p}_0(\Omega, \omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{1,p}_0(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).
In this paper we use the following results.

Theorem 2.1. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega, \omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, \omega)$ such that* *(i)* $u_{m_k}(x) \to u(x)$, $m_k \to \infty$ *a.e. on* Ω*;*

(ii) $|u_{m_k}(x)| \le \Phi(x)$ *a.e. on* Ω.

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [9]. □

Theorem 2.2. *(The weighted Sobolev inequality) Let* Ω *be an open bounded set in* $\mathbb{R}^n$ *and* $\omega \in A_p$ $(1 < p < \infty)$*. There exist constants* C_Ω *and* δ *positive such that for all* $u \in W_0^{1,p}(\Omega, \omega)$ *and all* k *satisfying* $1 \le k \le n/(n-1) + \delta$*,*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \| \, |\nabla u| \, \|_{L^p(\Omega,\omega)}, \tag{2.2}$$

where C_Ω *depends only on* n, p*, the* A_p*-constant* $C(p,\omega)$ *of* ω *and the diameter of* Ω*.*

Proof. Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [8]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega, \omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega, \omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega, \omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

Lemma 2.3. *Let* $1 < p < \infty$*.*

(a) There exists a constant $\alpha_p > 0$ *such that*

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \le \alpha_p \, |x-y|(|x|+|y|)^{p-2}, \forall x, y \in \mathbb{R}^n;$$

(b) There exist two positive constants β_p*,* γ_p *such that for every* $x, y \in \mathbb{R}^n$

$$\beta_p \, (|x|+|y|)^{p-2}|x-y|^2 \le \langle |x|^{p-2}x - |y|^{p-2}y, x-y \rangle \le \gamma_p \, (|x|+|y|)^{p-2}|x-y|^2,$$

where $\langle ., . \rangle$ *denotes the Euclidian scalar product in* $\mathbb{R}^n$*.*

Proof. See [5], Proposition 17.2 and Proposition 17.3. □

Definition 2.3. We denote by $X = W_0^{1,r}(\Omega, \omega_3) \cap W_0^{2,p}(\Omega, \omega_1)$ with the norm

$$\|u\|_X = \|\Delta u\|_{L^p(\Omega,\omega_1)} + \| \, |\nabla u| \, \|_{L^r(\Omega,\omega_3)}.$$

Definition 2.4. We say that an element $u \in X = W_0^{1,r}(\Omega, \omega_3) \cap W_0^{2,p}(\Omega, \omega_1)$ is a (weak) solution of problem (P) if

$$\begin{aligned}
&\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\, \omega_1\, dx + \int_\Omega |\Delta u|^{q-2}\Delta u\Delta\varphi\, \omega_2\, dx \\
&+ \int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla\varphi\rangle\, \omega_3\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\, \omega_4\, dx + \int_\Omega \mathcal{H}(x,u,\nabla u)\, \varphi\, \omega_5\, dx \\
&= \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\, dx,
\end{aligned}$$

for all $\varphi \in X$.

Remark 2.4. (a) If $\dfrac{\omega_2}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ $(2 \le q < p < \infty)$ then there exits a constant $M_1 > 0$ such that

$$\|u\|_{L^q(\Omega,\omega_2)} \le M_1\, \|u\|_{L^p(\Omega,\omega_1)},$$

where $M_1 = \|\omega_2/\omega_1\|^{1/q}_{L^{p/(p-q)}(\Omega,\omega_1)}$. In fact, by Hölder's inequality,

$$\begin{aligned}
\|u\|^q_{L^q(\Omega,\omega_2)} &= \int_\Omega |u|^q\, \omega_2\, dx = \int_\Omega |u|^q\, \frac{\omega_2}{\omega_1}\, \omega_1\, dx \\
&\le \left(\int_\Omega |u|^{q\,p/q}\omega_1\, dx\right)^{q/p} \left(\int_\Omega \left(\frac{\omega_2}{\omega_1}\right)^{p/(p-q)} \omega_1\, dx\right)^{(p-q)/p} \\
&= \|u\|^q_{L^p(\Omega,\omega_1)}\, \|\omega_2/\omega_1\|_{L^{p/(p-q)}(\Omega,\omega_1)}.
\end{aligned}$$

(b) Analogously, if $\dfrac{\omega_4}{\omega_3} \in L^{r/(r-s)}(\Omega, \omega_3)$ and $\dfrac{\omega_5}{\omega_3} \in L^{r/(r-z)}(\Omega, \omega_3)$ (with $1 < s, z < r < \infty$), there exist constants $M_2 > 0$ and $M_3 > 0$ (respectively), $M_2 = \|\omega_4/\omega_3\|^{1/s}_{L^{r/(r-s)}(\Omega,\omega_3)}$ and $M_3 = \|\omega_5/\omega_3\|^{1/z}_{L^{r/(r-z)}(\Omega,\omega_3)}$ such that

$$\|u\|_{L^s(\Omega,\omega_4)} \le M_2\, \|u\|_{L^r(\Omega,\omega_3)},$$

$$\|u\|_{L^z(\Omega,\omega_5)} \le M_3\, \|u\|_{L^r(\Omega,\omega_3)}.$$

3 Proof of Theorem 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3.1. *Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*
(a) For each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$;
(b) If the operator A is strictly monotone, then equation $A\,u = T$ is uniquely solvable in X.

Proof. See Theorem 26.A in [20]. □

To prove Theorem 1.1, we define $\mathbf{B}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4}, \mathbf{B_5} : X \times X \to \mathbb{R}$ and $\mathbf{T} : X \to \mathbb{R}$ by

$$\begin{aligned}
\mathbf{B}(u,\varphi) &= \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B_3}(u,\varphi) + \mathbf{B_4}(u,\varphi) + \mathbf{B_5}(u,\varphi),\\
\mathbf{B_1}(u,\varphi) &= \int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\, \omega_1\, dx,\\
\mathbf{B_2}(u,\varphi) &= \int_\Omega |\Delta u|^{q-2}\Delta u \Delta\varphi\, \omega_2\, dx,\\
\mathbf{B_3}(u,\varphi) &= \int_\Omega \langle \mathcal{A}(x,u,\nabla u), \nabla\varphi\rangle\, \omega_3\, dx,\\
\mathbf{B_4}(u,\varphi) &= \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\, \omega_4\, dx,\\
\mathbf{B_5}(u,\varphi) &= \int_\Omega \mathcal{H}(x,u,\nabla u)\, \varphi\, \omega_5\, dx,\\
\mathbf{T}(\varphi) &= \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^{n} \int_\Omega f_j\, D_j\varphi\, dx.
\end{aligned}$$

Then $u \in X$ is a (weak) solution to problem (P) if

$$\begin{aligned}
\mathbf{B}(u,\varphi) &= \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B_3}(u,\varphi) + \mathbf{B_4}(u,\varphi) + \mathbf{B_5}(u,\varphi)\\
&= \mathbf{T}(\varphi),
\end{aligned}$$

for all $\varphi \in X$.

Step 1. For $j = 1, ..., n$ we define the operator $F_j : X \to L^{r'}(\Omega, \omega_3)$ as

$$(F_j u)(x) = \mathcal{A}_j(x, u(x), \nabla u(x)).$$

We now show that the operator F_j is bounded and continuous.

(i) Using (H4) and Theorem 2.2 (with $k = 1$, since $\omega_3 \in A_r$) we obtain

$$\begin{aligned}
&\|F_j u\|^{r'}_{L^{r'}(\Omega,\omega_3)}\\
&= \int_\Omega |F_j u(x)|^{r'} \omega_3\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{r'} \omega_3\, dx \\
&\le \int_\Omega \left(K_1 + h_1|u|^{r/r'} + h_2|\nabla u|^{r/r'}\right)^{r'} \omega_3\, dx \\
&\le C_r \int_\Omega \left[(K_1^{r'} + h_1^{r'}|u|^r + h_2^{r'}|\nabla u|^r)\omega_3\right] dx \\
&= C_r\left[\int_\Omega K_1^{r'}\,\omega_3\, dx + \int_\Omega h_1^{r'}|u|^r\,\omega_3\, dx + \int_\Omega h_2^{r'}|\nabla u|^r \omega_3\, dx\right] \\
&\le C_r\left(\|K_1\|^{r'}_{L^{r'}(\Omega,\omega_3)} + \|h_1\|^{r'}_{L^\infty(\Omega)}\|u\|^r_{L^r(\Omega,\omega_3)} + \|h_2\|^{r'}_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^r_{L^r(\Omega,\omega_3)}\right) \\
&\le C_r\left(\|K_1\|^{r'}_{L^{r'}(\Omega,\omega_3)} + \|h_1\|^{r'}_{L^\infty(\Omega)} C^r_\Omega\|\,|\nabla u|\,\|^r_{L^r(\Omega,\omega_3)} + \|h_2\|^{r'}_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^r_{L^r(\Omega,\omega_3)}\right) \\
&\le C_r\left(\|K_1\|^{r'}_{L^{r'}(\Omega,\omega_3)} + (C^r_\Omega\,\|h_1\|^{r'}_{L^\infty(\Omega)} + \|h_2\|^{r'}_{L^\infty(\Omega)})\|u\|^r_X\right),
\end{aligned}
\tag{3.1}
$$

where the constant C_r depends only on r. Therefore, in (3.1) we obtain

$$
\begin{aligned}
&\|F_j u\|_{L^{r'}(\Omega,\omega_3)} \\
&\le C_r^{1/r'}\left(\|K_1\|_{L^{r'}(\Omega,\omega_3)} + (C_\Omega^{r-1}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\,\|u\|_X^{r-1}\right).
\end{aligned}
$$

(ii) Let $u_m \to u$ in X as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{r'}(\Omega,\omega_3)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m \to u$ in X, then $|\nabla u_m| \to |\nabla u|$ in $L^r(\Omega,\omega_3)$. Using Theorem 2.1 (since $\omega_3 \in A_r$), there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_1 \in L^r(\Omega,\omega_3)$ such that

$$
\begin{aligned}
&u_{m_k}(x) \to u(x) \text{ a.e. in } \Omega, \\
&D_j u_{m_k}(x) \to D_j u(x) \text{ a.e. in } \Omega, \\
&|\nabla u_{m_k}(x)| \le \Phi_1(x) \text{ a.e. in } \Omega.
\end{aligned}
\tag{3.2}
$$

By Theorem 2.2 (with $k = 1$, since $\omega_3 \in A_r$) we have

$$
\|u_{m_k}\|_{L^r(\Omega,\omega_3)} \le C_\Omega \|\,|\nabla u_{m_k}|\,\|_{L^r(\Omega,\omega_3)} \le C_\Omega \|\Phi_1\|_{L^r(\Omega,\omega_3)}, \tag{3.3}
$$

$$
\|u\|_{L^r(\Omega,\omega_3)} \le C_\Omega \|\,|\nabla u|\,\|_{L^r(\Omega,\omega_3)} \le C_\Omega \|\Phi_1\|_{L^r(\Omega,\omega_3)}. \tag{3.4}
$$

Next, applying (H4), (3.2), (3.3) and (3.4) we obtain

$$
\begin{aligned}
&\|F_j u_{m_k} - F_j u\|^{r'}_{L^{r'}(\Omega,\omega_3)} \\
&= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{r'} \omega_3 \, dx \\
&= \int_\Omega |\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{A}_j(x, u, \nabla u)|^{r'} \, \omega_3 \, dx \\
&\le C_r \int_\Omega \left(|\mathcal{A}_j(x, u_{m_k}, \nabla u_{m_k})|^{r'} + |\mathcal{A}_j(x, u, \nabla u)|^{r'} \right) \omega_3 \, dx \\
&\le C_r \left[\int_\Omega \left(K_1 + h_1 |u_{m_k}|^{r/r'} + h_2 |\nabla u_{m_k}|^{r/r'} \right)^{r'} \omega_3 \, dx \right. \\
&\left. + \int_\Omega \left(K_1 + h_1 |u|^{r/r'} + h_2 |\nabla u|^{r/r'} \right)^{r'} \omega_3 \, dx \right] \\
&\le C_r \left[\int_\Omega K_1^{r'} \omega_3 \, dx + \int_\Omega h_1^{r'} |u_{m_k}|^r \omega_3 \, dx + \int_\Omega h_2^{r'} |\nabla u_{m_k}|^r \omega_3 \, dx \right. \\
&\left. + \int_\Omega K_1^{r'} \omega_3 \, dx + \int_\Omega h_1^{r'} |u|^r \, \omega_3 \, dx + \int_\Omega h_2^{r'} |\nabla u|^r \, \omega_3 \, dx \right] \\
&\le 2\, C_r \left[\|K_1\|^{r'}_{L^{r'}(\Omega,\omega_3)} + (C^r_\Omega \|h_1\|^{r'}_{L^\infty(\Omega)} + \|h_2\|^{r'}_{L^\infty(\Omega)}) \|\Phi_1\|^r_{L^r(\Omega,\omega_3)} \right].
\end{aligned}
$$

By condition (H1), we have

$$
F_j u_{m_k}(x) = \mathcal{A}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \rightarrow \mathcal{A}_j(x, u(x), \nabla u(x)) = F_j u(x),
$$

as $m_k \rightarrow +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$
\|F_j u_{m_k} - F_j u\|_{L^{r'}(\Omega,\omega_3)} \rightarrow 0,
$$

that is,

$$
F_j u_{m_k} \rightarrow F_j u \;\text{ in }\; L^{r'}(\Omega, \omega_3).
$$

We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$
F_j u_m \rightarrow F_j u \;\text{ in }\; L^{r'}(\Omega, \omega_3). \tag{3.5}
$$

Step 2. We define the operator $G_j : X \rightarrow L^{s'}(\Omega, \omega_4)$ $(j = 1, ..., n)$ by

$$
(G_j u)(x) = \mathcal{B}_j(x, u(x), \nabla u(x)).
$$

This operator is continuous and bounded. In fact,

(i) Using (H8), Theorem 2.2 (since $\omega_3 \in A_r$) and Remark 2.4(b) we obtain

$$
\begin{aligned}
&\|G_j u\|^{s'}_{L^{s'}(\Omega,\omega_4)}\\
&= \int_\Omega |G_j u(x)|^{s'} \omega_4 \, dx\\
&= \int_\Omega |\mathcal{B}_j(x,u,\nabla u)|^{s'} \omega_4 \, dx\\
&\leq \int_\Omega \left(K_2 + g_1|u|^{s/s'} + g_2|\nabla u|^{s/s'}\right)^{s'} \omega_4 \, dx\\
&\leq C_s \int_\Omega \left[(K_2^{s'} + g_1^{s'}|u|^s + g_2^{s'}|\nabla u|^s)\omega_4 \right] dx\\
&= C_s \left[\int_\Omega K_2^{s'} \omega_4 \, dx + \int_\Omega g_1^{s'}|u|^s \omega_4 \, dx + \int_\Omega g_2^{s'}|\nabla u|^s \omega_4 \, dx \right]\\
&\leq C_s \left(\|K_2\|^{s'}_{L^{s'}(\Omega,\omega_4)} + \|g_1\|^{s'}_{L^\infty(\Omega)} \|u\|^s_{L^s(\Omega,\omega_4)} + \|g_2\|^{s'}_{L^\infty(\Omega)} \| \, |\nabla u| \, \|^s_{L^s(\Omega,\omega_4)} \right)\\
&\leq C_s \Big(\|K_2\|^{s'}_{L^{s'}(\Omega,\omega_4)} + \|g_1\|^{s'}_{L^\infty(\Omega)} \, M_2^s \, \|u\|^s_{L^r(\Omega,\omega_3)}\\
&+ M_2^s \|g_2\|^{s'}_{L^\infty(\Omega)} \| \, |\nabla u| \, \|^s_{L^r(\Omega,\omega_3)} \Big)\\
&\leq C_s \Big(\|K_2\|^{s'}_{L^{s'}(\Omega,\omega_4)} + \|g_1\|^{s'}_{L^\infty(\Omega)} \, C_\Omega^s \, M_2^s \, \| \, |\nabla u| \, \|^s_{L^r(\Omega,\omega_3)}\\
&+ M_2^s \|g_2\|^{s'}_{L^\infty(\Omega)} \| \, |\nabla u| \, \|^s_{L^r(\Omega,\omega_3)} \Big)\\
&\leq C_s \left(\|K_2\|^{s'}_{L^{s'}(\Omega,\omega_4)} + (C_\Omega^s \, M_2^s \|g_1\|^{s'}_{L^\infty(\Omega)} + M_2^s \|g_2\|^{s'}_{L^\infty(\Omega)}) \|u\|^s_X \right), \qquad (3.6)
\end{aligned}
$$

where the C_s depends only on s. Therefore, in (3.6), we obtain

$$
\begin{aligned}
&\|G_j u\|_{L^{s'}(\Omega,\omega_4)}\\
&\leq C_s^{1/s'} \left(\|K_2\|_{L^{s'}(\Omega,\omega_4)} + M_2^{s-1} (C_\Omega^{s-1} \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}) \, \|u\|^{s-1}_X \right). \qquad (3.7)
\end{aligned}
$$

(ii) Let $u_m \rightarrow u$ in X as $m \rightarrow \infty$. We need to show that $G_j u_m \rightarrow G_j u$ in $L^{s'}(\Omega, \omega_4)$. We will apply the Lebesgue Dominated Theorem. If $u_m \rightarrow u$ in X, then $|\nabla u_m| \rightarrow |\nabla u|$ in $L^r(\Omega, \omega_3)$.

Next, applying (H8), Theorem 2.2, Remark 2.4(b), (3.2), (3.3) and (3.4) we obtain

$$
\begin{aligned}
\|G_j u_{m_k} - G_j u\|^{s'}_{L^{s'}(\Omega,\omega_4)} &= \int_\Omega |G_j u_{m_k}(x) - G_j u(x)|^{s'} \omega_4 \, dx\\
&= \int_\Omega |\mathcal{B}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{B}_j(x,u,\nabla u)|^{s'} \, \omega_4 \, dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C_s \int_\Omega \Big(|\mathcal{B}_j(x, u_{m_k}, \nabla u_{m_k})|^{s'} + |\mathcal{B}_j(x, u, \nabla u)|^{s'} \Big) \omega_4\, dx \\
&\le C_s \Bigg[\int_\Omega \Big(K_2 + g_1 |u_{m_k}|^{s/s'} + g_2 |\nabla u_{m_k}|^{s/s'} \Big)^{s'} \omega_4\, dx \\
&+ \int_\Omega \Big(K_2 + g_1 |u|^{s/s'} + g_2 |\nabla u|^{s/s'} \Big)^{s'} \omega_4\, dx \Bigg] \\
&\le C_s \Bigg[\int_\Omega K_2^{s'} \omega_4\, dx + \|g_1\|^{s'}_{L^\infty(\Omega)} \int_\Omega |u_{m_k}|^s \omega_4\, dx + \|g_2\|^{s'}_{L^\infty(\Omega)} \int_\Omega |\nabla u_{m_k}|^s \omega_4\, dx \\
&+ \int_\Omega K_2^{s'} \omega_4\, dx + \|g_1\|^{s'}_{L^\infty(\Omega)} \int_\Omega |u|^s \omega_4\, dx + \|g_2\|^{s'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^s \omega_4\, dx \Bigg] \\
&\le C_s \Bigg[\int_\Omega K_2^{s'} \omega_4\, dx + \|g_1\|^{s'}_{L^\infty(\Omega)} M_2^s \|u_{m_k}\|^s_{L^r(\Omega,\omega_3)} + \|g_2\|^{s'}_{L^\infty(\Omega)} M_2^s \|\, |\nabla u_{m_k}|\, \|^s_{L^r(\Omega,\omega_3)} \\
&+ \int_\Omega K_2^{s'} \omega_4\, dx + \|g_1\|^{s'}_{L^\infty(\Omega)} M_2^s \|u\|^s_{L^r(\Omega,\omega_3)} + \|g_2\|^{s'}_{L^\infty(\Omega)} M_2^s \|\, |\nabla u|\, \|^s_{L^r(\Omega,\omega_3)} \\
&\le 2C_s \Bigg[\|K_2\|^{s'}_{L^{s'}(\Omega,\omega_4)} + \|g_1\|^{s'}_{L^\infty(\Omega)} C_\Omega^s M_2^s \|\Phi_1\|^s_{L^r(\Omega,\omega_3)} + \|g_2\|^{s'}_{L^\infty(\Omega)} M_2^s \|\Phi_1\|^s_{L^r(\Omega,\omega_3)} \\
&= 2\, C_s \Big(\|K_2\|^{s'}_{L^{s'}(\Omega,\omega_4)} + M_2^s \big(C_\Omega^s \|g_1\|^{s'}_{L^\infty(\Omega)} + \|g_2\|^{s'}_{L^\infty(\Omega)} \big) \|\Phi_1\|_{L^r(\Omega,\omega_3)} \Bigg].
\end{aligned}
$$

By condition (H5), we have

$$G_j u_{m_k}(x) = \mathcal{B}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{B}_j(x, u(x), \nabla u(x)) = G_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|G_j u_{m_k} - G_j u\|_{L^{s'}(\Omega,\omega_4)} \to 0,$$

that is,

$$G_j u_{m_k} \to G_j u \ \text{ in } \ L^{s'}(\Omega, \omega_4).$$

We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$G_j u_m \to G_j u \ \text{ in } \ L^{s'}(\Omega, \omega_4). \tag{3.8}$$

Step 3. We define the operator $F : X \to L^{p'}(\Omega, \omega_1)$ by

$$(Fu)(x) = |\Delta u(x)|^{p-2} \Delta u(x).$$

We now show that operator F is bounded and continuous.

(i) We have

$$\begin{aligned}
\|Fu\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_1u(x)|^{p'}\,\omega_1\,dx \\
&= \int_\Omega \left||\Delta u|^{p-2}\,\Delta u\right|^{p'}\,\omega_1\,dx \\
&= \int_\Omega |\Delta u|^p\,\omega_1\,dx \\
&= \|\Delta u\|^p_{L^p(\Omega,\omega_1)} \\
&\le \|u\|^p_X.
\end{aligned} \tag{3.9}$$

Therefore, in (3.9), we obtain

$$\|F\,u\|_{L^{p'}(\Omega,\omega_1)} \le \|u\|^{p-1}_X, \tag{3.10}$$

and hence the boundedness.

(ii) Let $u_m \to u$ in X as $m \to 0$. We need to show that $Fu_m \to Fu$ in $L^{p'}(\Omega,\omega_1)$. If $u_m \to u$ in X then $\Delta u_m \to \Delta u$ in $L^p(\Omega,\omega_1)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_2 \in L^p(\Omega,\omega_1)$ such that

$$\Delta u_{m_k}(x) \to \Delta u(x) \text{ a.e. in } \Omega, \tag{3.11}$$

$$|\Delta u_{m_k}(x)| \le \Phi_2(x) \text{ a.e. in } \Omega. \tag{3.12}$$

Now, since $p > 2$, using (3.11), (3.12), $a = p/p' = p-1$ and $a' = (p-1)/(p-2)$, there exists a constant $\alpha_p > 0$ (by Lemma 2.3(a)) such that

$$\begin{aligned}
&\|Fu_{m_k} - Fu\|^{p'}_{L^{p'}(\Omega,\omega_1)} \\
&= \int_\Omega |Fu_{m_k} - Fu|^{p'}\omega_1\,dx \\
&= \int_\Omega \left||\Delta u_{m_k}|^{p-2}\Delta u_{m_k} - |\Delta u|^{p-2}\Delta u\right|^{p'}\,\omega_1\,dx \\
&\le \int_\Omega \left[\alpha_p\,|\Delta u_{m_k} - \Delta u|\,(\,|\Delta u_{m_k}| + |\Delta u|\,)^{p-2}\right]^{p'}\,\omega_1\,dx \\
&\le \alpha_p^{p'} \int_\Omega |\Delta u_{m_k} - \Delta u|^{p'}(2\Phi_2)^{(p-2)p'}\,\omega_1\,dx \\
&= 2^{(p-2)p'}\alpha_p^{p'} \int_\Omega |\Delta u_{m_k} - \Delta u|^{p'}\Phi_2^{(p-2)p'}\,\omega_1\,dx \\
&\le 2^{(p-2)p'}\alpha_p^{p'}\left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{p'a}\omega_1\,dx\right)^{1/a}\left(\int_\Omega \Phi_2^{(p-2)p'a'}\,\omega_1\,dx\right)^{1/a'}
\end{aligned}$$

$$
\begin{aligned}
&= 2^{(p-2)p'}\alpha_p^{p'}\left(\int_\Omega |\Delta u_{m_k}-\Delta u|^p\,\omega_1\,dx\right)^{p'/p}\left(\int_\Omega \Phi_2^p\,\omega_1\,dx\right)^{(p-2)/(p-1)}\\
&= 2^{(p-2)p'}\alpha_p^{p'}\|\Delta u_{m_k}-\Delta u\|_{L^p(\Omega,\omega_1)}^{p'}\|\Phi_2\|_{L^p(\Omega,\omega_1)}^{p'(p-2)}\\
&\le 2^{(p-2)p'}\alpha_p^{p'}\|u_{m_k}-u\|_X^{p'}\|\Phi_2\|_{L^p(\Omega,\omega_1)}^{p'(p-2)}.
\end{aligned}
$$

Hence,

$$
\|Fu_{m_k}-Fu\|_{L^{p'}(\Omega,\omega_1)}\le 2^{p-2}\,\alpha_p\,\|u_{m_k}-u\|_X\,\|\Phi_2\|_{L^p(\Omega,\omega_1)}^{p-2}.
$$

Therefore (since $2<p<\infty$), we obtain $\|Fu_{m_k}-Fu\|_{L^{p'}(\Omega,\omega_1)}\to 0$, that is,

$$
Fu_{m_k}\to Fu\ \text{ in }\ L^{p'}(\Omega,\omega_1).
$$

By the Convergence Principle in Banach spaces (see Proposition 10.13 in [19], we have

$$
Fu_m\to Fu\ \text{ in }\ L^{p'}(\Omega,\omega_1). \tag{3.13}
$$

Step 4. Define the operator $G : X\to L^{q'}(\Omega,\omega_2)$, $(Gu)(x)=|\Delta u(x)|^{q-2}\Delta u(x)$. We also have that the operator G is continuous and bounded. In fact:

(i) If $q>2$, we have by Remark 2.4 (a)

$$
\begin{aligned}
\|Gu\|_{L^{q'}(\Omega,\omega_2)}^{q'} &= \int_\Omega \big||\Delta u|^{q-2}\,\Delta u\big|^{q'}\,\omega_2\,dx=\int_\Omega |\Delta u|^q\,\omega_2\,dx\\
&= \|\Delta u\|_{L^q(\Omega,\omega_2)}^q\\
&\le M_1^q\|\Delta u\|_{L^p(\Omega,\omega_1)}^q\\
&\le M_1^q\,\|u\|_X^q.
\end{aligned}
$$

Hence, $\|Gu\|_{L^{q'}(\Omega,\omega_2)}\le M_1^{q-1}\|u\|_X^{q-1}$.

(ii) Now using (3.11), (3.12), Remark 2.4(a), $b=q/q'=q-1$ and $b'=(q-1)/(q-2)$ (if $q>2$), there exists a constant $\alpha_q>0$ (by Lemma 2.3(a)) such that

$$
\begin{aligned}
&\|Gu_{m_k}-Gu\|_{L^{q'}(\Omega,\omega_2)}^{q'}=\int_\Omega |\,Gu_{m_k}-Gu|^{q'}\,\omega_2\,dx\\
&=\int_\Omega \left||\Delta u_{m_k}|^{q-2}\,\Delta u_{m_k}-|\,\Delta u|^{q-2}\,\Delta u\right|^{q'}\,\omega_2\,dx\\
&\le \int_\Omega \left[\alpha_q\,|\Delta u_{m_k}-\Delta u|\,(\,|\Delta u_{m_k}|+|\Delta u|)^{(q-2)}\right]^{q'}\,\omega_2\,dx\\
&\le \alpha_q^{q'}\int_\Omega |\,\Delta u_{m_k}-\Delta u|^{q'}\,(2\,\Phi_2)^{(q-2)\,q'}\,\omega_2\,dx
\end{aligned}
$$

$$\begin{aligned}
&\leq 2^{(q-2)q'}\alpha_q^{q'}\left(\int_\Omega |\Delta u_{m_k}-\Delta u|^{q'b}\omega_2\,dx\right)^{1/b}\left(\int_\Omega \Phi_2^{(q-2)q'b'}\omega_2\,dx\right)^{1/b'}\\
&= \alpha_q^{q'}\,2^{(q-2)q'}\left(\int_\Omega |\Delta u_{m_k}-\Delta u|^{q}\,\omega_2\,dx\right)^{q'/q}\left(\int_\Omega \Phi_2^{q}\,\omega_2\,dx\right)^{(q-2)/(q-1)}\\
&= \alpha_q^{q'}\,2^{(q-2)q'}\|\Delta u_{m_k}-\Delta u\|^{q'}_{L^q(\Omega,\omega_2)}\|\Phi_2\|^{q'(q-2)}_{L^q(\Omega,\omega_2)}\\
&\leq \alpha_q^{q'}\,2^{(q-2)q'}M_1^{q'}\|\Delta u_{m_k}-\Delta u\|^{q'}_{L^p(\Omega,\omega_1)}M_1^{q'(q-2)}\|\Phi_2\|^{q'(q-2)}_{L^p(\Omega,\omega_1)}\\
&\leq \alpha_q^{q'}\,2^{(q-2)q'}M_1^{q}\|u_{m_k}-u\|^{q'}_X\|\Phi_2\|^{q'(q-2)}_{L^p(\Omega,\omega_1)}.
\end{aligned}$$

Hence, $\|Gu_{m_k}-Gu\|_{L^{q'}(\Omega,\omega_2)}\leq 2^{q-2}\alpha_q\,M_1^{q-1}\|\Phi_2\|^{q-2}_{L^p(\Omega,\omega_1)}\,\|u_{m_k}-u\|_X$.

In the case $q=2$ we have $(Gu)(x)=\Delta u(x)$. Hence,

$$\begin{aligned}
&\|Gu\|_{L^2(\Omega,\omega_2)}=\|\Delta u\|_{L^2(\Omega,\omega_2)}\leq M_1\|\Delta u\|_{L^p(\Omega,\omega_1)}\leq M_1\|u\|_X,\\
&\|Gu_{m_k}-Gu\|_{L^2(\Omega,\omega_2)}\leq M_1\|\Delta u_{m_k}-\Delta u\|_{L^p(\Omega,\omega_1)}\leq M_1\|u_{m_k}-u\|_X.
\end{aligned}$$

Therefore (for $2\leq q<\infty$), we obtain $\|Gu_{m_k}-Gu\|_{L^{q'}(\Omega,\omega_2)}\to 0$, that is, $Gu_{m_k}\to Gu$ in $L^{q'}(\Omega,\omega_2)$. By the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]), we have

$$Gu_m\to Gu \text{ in } L^{q'}(\Omega,\omega_2). \tag{3.14}$$

Step 5. We define the operator $H: X\to L^{z'}(\Omega,\omega_5)$ by

$$(Hu)(x)=\mathcal{H}(x,u(x),\nabla u(x)).$$

We also have that the operator H is continuous and bounded. In fact,

(i) Using $(H12)$, Theorem 2.2 (since $\omega_3\in A_r$) and Remark 2.4(b) we obtain

$$\begin{aligned}
&\|Hu\|^{z'}_{L^{z'}(\Omega,\omega_5)}\\
&=\int_\Omega |Hu|^{z'}\,\omega_5\,dx\\
&=\int_\Omega |\mathcal{H}(x,u,\nabla u)|^{z'}\,\omega_5\,dx\\
&\leq \int_\Omega\left(K_3+h_3|u|^{z/z'}+h_4|\nabla u|^{z/z'}\right)^{z'}\omega_5\,dx\\
&\leq C_z\int_\Omega (K_3^{z'}+h_3^{z'}|u|^z+h_4^{z'}|\nabla u|^z)\omega_5\,dx\\
&\leq C_z\int_\Omega K_3^{z'}\,\omega_5\,dx+\|h_3\|^{z'}_{L^\infty(\Omega)}\int_\Omega |u|^z\,\omega_5\,dx+\|h_4\|^{z'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^z\omega_5\,dx\bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq C_z\bigg(\|K_3\|^{z'}_{L^{z'}(\Omega,\omega_5)} + \|h_3\|^{z'}_{L^\infty(\Omega)} M_3^z \|u\|^z_{L^r(\Omega,\omega_3)}\\
&+\|h_4\|^{z'}_{L^\infty(\Omega)} M_3^z \|\,|\nabla u|\,\|^z_{L^r(\Omega,\omega_3)}\bigg)\\
&\leq C_z\bigg(\|K_3\|^{z'}_{L^{z'}(\Omega,\omega_5)} + M_3^z(C_\Omega^z\|h_3\|^{z'}_{L^\infty(\Omega)} + \|h_4\|^{z'}_{L^\infty(\Omega)})\,\|u\|^z_X\bigg),
\end{aligned}
\tag{3.15}
$$

where the constant C_z depends only on z. Hence, in (3.15), we obtain

$$
\|Hu\|_{L^{z'}(\Omega,\omega_5)} \leq C_z^{1/z'}\bigg[\|K_3\|_{L^{z'}(\Omega,\omega_5)} + M_3^{z-1}(C_\Omega^{z-1}\|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)})\|u\|_X^{z-1}\bigg].
$$

(ii) Applying (H9), (H12), Remark 2.4(b), (3.2), (3.3) and (3.4) by the same argument used in Step 1(ii), we obtain analogously, if $u_m \to u$ in X then

$$
Hu_m \to Hu, \text{ in } L^{z'}(\Omega,\omega_5). \tag{3.16}
$$

Step 6. Since $\dfrac{f_j}{\omega_3} \in L^{r'}(\Omega,\omega_3)$ $(j = 0,1,...,n)$ then $\mathbf{T} \in X^*$. Moreover, by Theorem 2.2 (with $k = 1$, since $\omega_3 \in A_r$) we have

$$
\begin{aligned}
|\mathbf{T}(\varphi)| &\leq \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx\\
&= \int_\Omega \frac{|f_0|}{\omega_3}|\varphi|\omega_3\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_3}|D_j\varphi|\,\omega_3\,dx\\
&\leq \|f_0/\omega_3\|_{L^{r'}(\Omega,\omega_3)}\|\varphi\|_{L^r(\Omega,\omega_3)} + \bigg(\sum_{j=1}^n \|f_j/\omega_3\|_{L^{r'}(\Omega,\omega_3)}\bigg)\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&\leq C_\Omega\|f_0/\omega_3\|_{L^{r'}(\Omega,\omega_3)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&+ \bigg(\sum_{j=1}^n \|f_j/\omega_3\|_{L^{r'}(\Omega,\omega_3)}\bigg)\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&\leq \bigg(C_\Omega\,\|f_0/\omega_3\|_{L^{r'}(\Omega,\omega_3)} + \sum_{j=1}^n \|f_j/\omega_3\|_{L^{r'}(\Omega,\omega_3)}\bigg)\|\varphi\|_X.
\end{aligned}
$$

Moreover, we also have

$$
\begin{aligned}
|\mathbf{B}(u,\varphi)| &\leq |\mathbf{B_1}(u,\varphi)| + |\mathbf{B_2}(u,\varphi)| + |\mathbf{B_3}(u,\varphi)| + |\mathbf{B_4}(u,\varphi)| + |\mathbf{B_5}(u,\varphi)|\\
&\leq \int_\Omega |\mathcal{A}(x,u,\nabla u)||\nabla\varphi|\,\omega_3\,dx + \int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_4\,dx\\
&+ \int_\Omega |\Delta u|^{p-1}|\Delta\varphi|\,\omega_1\,dx + \int_\Omega |\Delta u|^{q-1}|\Delta\varphi|\,\omega_2\,dx
\end{aligned}
$$

$$+ \quad \int_\Omega |\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\omega_5\,dx. \tag{3.17}$$

In (3.17) we have:

(i) By (H4) and Theorem 2.2,

$$\begin{aligned}
&\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_3\,dx\\
&\leq \int_\Omega \left(K_1 + h_1|u|^{r/r'} + h_2|\nabla u|^{r/r'}\right)|\nabla\varphi|\,\omega_3\,dx\\
&\leq \|K_1\|_{L^{r'}(\Omega,\omega_3)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{r/r'}_{L^r(\Omega,\omega_3)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&+\|h_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{r/r'}_{L^r(\Omega,\omega_3)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&\leq \left(\|K_1\|_{L^{r'}(\Omega,\omega_3)} + (C_\Omega^{r-1}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{r-1}\right)\|\varphi\|_X.
\end{aligned}$$

(ii) By (H8), Theorem 2.2 (with $k = 1$, since $\omega_4 \in A_s$) and Remark 2.4(b)

$$\begin{aligned}
&\int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_4\,dx\\
&\leq \int_\Omega \left(K_2 + g_1|u|^{s/s'} + g_2|\nabla u|^{s/s'}\right)|\nabla\varphi|\,\omega_4\,dx\\
&\leq \|K_2\|_{L^{s'}(\Omega,\omega_4)}\|\,|\nabla\varphi|\,\|_{L^s(\Omega,\omega_4)} + \|g_1\|_{L^\infty(\Omega)}\|u\|^{s/s'}_{L^s(\Omega,\omega_4)}\|\,|\nabla\varphi|\,\|_{L^s(\Omega,\omega_4)}\\
&+\|g_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{s/s'}_{L^s(\Omega,\omega_4)}\|\,|\nabla\varphi|\,\|_{L^s(\Omega,\omega_4)}\\
&\leq M_2\|K_2\|_{L^{s'}(\Omega,\omega_4)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&+M_2^{s-1}\,C_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{s-1}_{L^r(\Omega,\omega_3)}\,M_2\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&+M_2^{s-1}\,\|g_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{s-1}_{L^r(\Omega,\omega_3)}\,M_2\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\omega_3)}\\
&\leq \left[M_2\|K_2\|_{L^{s'}(\Omega,\omega_4)} + \left(M_2^s\,C_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)} + M_2^s\|g_2\|_{L^\infty(\Omega)}\right)\|u\|_X^{s-1}\right]\|\varphi\|_X.
\end{aligned}$$

(iii) By (H12), Theorem 2.2 (since $\omega_3 \in A_r$) and Remark 2.4(b)

$$\begin{aligned}
&\int_\Omega |\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\omega_5\,dx\\
&\leq \int_\Omega \left(K_3 + h_3|u|^{z/z'} + h_4|\nabla u|^{z/z'}\right)|\varphi|\,\omega_5\,dx\\
&\leq \int_\Omega K_3\,|\varphi|\,\omega_5\,dx + \|h_3\|_{L^\infty(\Omega)}\int_\Omega |u|^{z/z'}|\varphi|\,\omega_5\,dx + \|h_4\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^{z/z'}|\varphi|\,\omega_5\,dx\\
&\leq \|K_3\|_{L^{z'}(\Omega,\omega_5)}\|\varphi\|_{L^z(\Omega,\omega_5)} + \|h_3\|_{L^\infty(\Omega)}\|u\|^{z/z'}_{L^z(\Omega,\omega_5)}\|\varphi\|_{L^z(\Omega,\omega_5)}\\
&+\|h_4\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{z/z'}_{L^z(\Omega,\omega_5)}\|\varphi\|_{L^z(\Omega,\omega_5)}
\end{aligned}$$

$$
\begin{aligned}
&\leq M_3\,\|K_3\|_{L^{z\,\prime}(\Omega)}\|\varphi\|_{L^r(\Omega,\omega_3)} + \|h_3\|_{L^\infty(\Omega)}M_3^{z-1}\|u\|^{z-1}_{L^r(\Omega,\omega_3)}M_3\|\varphi\|_{L^r(\Omega,\omega_3)}\\
&+\|h_4\|_{L^\infty(\Omega)}M_3^{z-1}\|\,|\nabla u|\,\|^{z-1}_{L^r(\Omega,\omega_3)}M_3\|\varphi\|_{L^r(\Omega,\omega_3)}\\
&\leq\left[M_3C_\Omega\,\|K_3\|_{L^{z\,\prime}(\Omega,\omega_5)} + M_3^z(C_\Omega^{z-1}\|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)})\|u\|_X^{z-1}\right]\|\varphi\|_X.
\end{aligned}
$$

(iv) We have

$$
\begin{aligned}
\int_\Omega |\Delta u|^{p-1}|\Delta\varphi|\,\omega_1\,dx &\leq \left(\int_\Omega |\Delta u|^{(p-1)\,p\,\prime}\omega_1\,dx\right)^{1/p\,\prime}\left(\int_\Omega |\Delta\varphi|^p\,\omega_1\,dx\right)^{1/p}\\
&= \|\Delta u\|^{p-1}_{L^p(\Omega,\omega_1)}\,\|\Delta\varphi\|_{L^p(\Omega,\omega_1)}\\
&\leq \|u\|_X^{p-1}\,\|\varphi\|_X;
\end{aligned}
$$

(v) By Remark 2.4(a),

$$
\begin{aligned}
\int_\Omega |\Delta u|^{q-1}|\Delta\varphi|\,\omega_2\,dx &\leq \left(\int_\Omega |\Delta u|^{(q-1)\,q\,\prime}\omega_2\,dx\right)^{1/q\,\prime}\left(\int_\Omega |\Delta\varphi|^q\,\omega_2\,dx\right)^{1/q}\\
&= \|\Delta u\|^{q-1}_{L^q(\Omega,\omega_2)}\,\|\Delta\varphi\|_{L^q(\Omega,\omega_2)}\\
&\leq M_1^{q-1}\|\Delta u\|^{q-1}_{L^p(\Omega,\omega_1)}\,M_1\,\|\Delta\varphi\|_{L^p(\Omega,\omega_1)}\\
&\leq M_1^q\,\|u\|_X^{q-1}\,\|\varphi\|_X.
\end{aligned}
$$

Hence, in (3.17) we obtain, for all $u,\varphi\in X$

$$
\begin{aligned}
|\mathbf{B}(u,\varphi)| \leq\; &\Big[\|K_1\|_{L^{r\,\prime}(\Omega,\omega_1)} + (C_\Omega^{r-1}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{r-1}\\
+\;& M_2\|K_2\|_{L^{s\,\prime}(\Omega,\omega_4)} + M_2^s(C_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)})\|u\|_X^{s-1}\\
+\;& M_3C_\Omega\|K_3\|_{L^{z\,\prime}(\Omega,\omega_5)} + M_3^z(C_\Omega^{z-1}\|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)})\|u\|_X^{z-1}\\
+\;& \|u\|_X^{p-1} + M_1^q\|u\|_X^{q-1}\Big]\|\varphi\|_X.
\end{aligned}
$$

Since $\mathbf{B}(u,.)$ is linear, for each $u\in X$, there exists a linear and continuous functional on X denoted by Au such that $(Au|\varphi) = \mathbf{B}(u,\varphi)$, for all $u,\ \varphi\in X$ (here $(f|x)$ denotes the value of the linear functional f at the point x). Moreover

$$
\begin{aligned}
\|Au\|_* \leq\; & \|K_1\|_{L^{r\,\prime}(\Omega,\omega_1)} + (C_\Omega^{r-1}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{r-1}\\
+\;& M_2\|K_2\|_{L^{s\,\prime}(\Omega,\omega_4)} + M_2^s(C_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)})\|u\|_X^{s-1}\\
+\;& M_3C_\Omega\|K_3\|_{L^{z\,\prime}(\Omega,\omega_5)} + M_3^z(C_\Omega^{z-1}\|h_3\|_{L^\infty(\Omega)} + \|h_4\|_{L^\infty(\Omega)})\|u\|_X^{z-1}\\
+\;& \|u\|_X^{p-1} + M_1^q\|u\|_X^{q-1},
\end{aligned}
$$

where $\|Au\|_* = \sup\{|(Au|\varphi)| = |\mathbf{B}(u,\varphi)| : \varphi\in X, \|\varphi\|_X = 1\}$ is the norm of operator Au.

Hence, we obtain the operator

$$A : X \longrightarrow X^*$$
$$u \mapsto Au.$$

Consequently, problem (P) is equivalent to the operator equation $Au = \mathbf{T},\ u \in X$.

Step 7. Using (H2), (H6), (H10) and Lemma 2.3(b), we obtain (for $u_1, u_2 \in X$, $u_1 \neq u_2$)

$$\begin{aligned}
&(Au_1 - Au_2|u_1 - u_2)\\
&= \mathbf{B}(u_1, u_1 - u_2) - \mathbf{B}(u_2, u_1 - u_2)\\
&= \int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\Big)\,\Delta(u_1 - u_2)\,\omega_1\,dx\\
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)\,\Delta(u_1 - u_2)\,\omega_2\,dx\\
&+ \int_\Omega \langle \mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2\nabla u_2), \nabla(u_1 - u_2)\rangle\,\omega_3\,dx\\
&+ \int_\Omega \langle \mathcal{B}(x, u_1, \nabla u_1) - \mathcal{B}(x, u_2, \nabla u_2), \nabla(u_1 - u_2)\rangle\,\omega_4\,dx\\
&+ \int_\Omega \big(\mathcal{H}(x, u_1, \nabla u_1) - \mathcal{H}(x, u_2, \nabla u_2)\big)(u_1 - u_2)\,\omega_5\,dx\\
&\geq \beta_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\,\omega_1\,dx\\
&+ \beta_q \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{q-2} |\Delta u_1 - \Delta u_2|^2\,\omega_2\,dx\\
&+ \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^r\,\omega_3\,dx + \theta_2 \int_\Omega |\nabla(u_1 - u_2)|^s\,\omega_4\,dx\\
&\geq \beta_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2} |\Delta u_1 - \Delta u_2|^2\,\omega_1\,dx + \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^r\,\omega_3\,dx\\
&\geq \beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2}|\Delta u_1 - \Delta u_2|^2\,\omega_1 + \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^r\,\omega_3\,dx\\
&= \beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^p\,\omega_1\,dx + \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^r\,\omega_3\,dx > 0.
\end{aligned}$$

Therefore, the operator A is strictly monotone. Moreover, we have by (H3), (H7) and (H11),

$$\begin{aligned}
(Au|u) &= \mathbf{B}(u, u) = \mathbf{B}_1(u, u) + \mathbf{B}_2(u, u) + \mathbf{B}_3(u, u) + \mathbf{B}_4(u, u) + \mathbf{B}_5(u, u)\\
&= \int_\Omega |\Delta u|^p \omega_1\,dx + \int_\Omega |\Delta u|^q \omega_2\,dx\\
&+ \int_\Omega \langle \mathcal{A}(x, u, \nabla u), \nabla u\rangle\,\omega_3\,dx + \int_\Omega \langle \mathcal{B}(x, u, \nabla u), \nabla u\rangle\,\omega_4\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega \mathcal{H}(x,u,\nabla u)\, u\, \omega_5\, dx \\
&\geq \int_\Omega |\Delta u|^p \omega_1\, dx + \int_\Omega |\nabla u|^q\, \omega_2\, dx \\
&+ \lambda_1 \int_\Omega |\nabla u|^r\, \omega_3\, dx + \lambda_2 \int_\Omega |\nabla u|^s\, \omega_4\, dx + \Lambda_2 \int_\Omega |u|^s\, \omega_4\, dx \\
&+ \lambda_3 \int_\Omega |\nabla u|^z\, \omega_5\, dx + \Lambda_3 \int_\Omega |u|^z \omega_5\, dx \\
&\geq \int_\Omega |\Delta u|^p \omega_1\, dx + \lambda_1 \int_\Omega |\nabla u|^r\, \omega_3\, dx \\
&\geq \gamma \left(\|\Delta u\|^p_{L^p(\Omega,\omega_1)} + \|\, |\nabla u|\, \|^r_{L^r(\Omega,\omega_3)} \right),
\end{aligned}
$$

where $\gamma = \min\{\lambda_1, 1\}$. Hence, since $2 < p < \infty$ and $1 < r < \infty$ we have

$$
\frac{(Au|u)}{\|u\|_X} \longrightarrow +\infty, \text{ as } \|u\|_X \longrightarrow +\infty,
$$

that is, A is coercive (using that $\lim_{t+a\to\infty} \frac{t^p + a^r}{t+a} = \infty$, with $t > 0$ and $a > 0$).

Step 8. We need to show that the operator A is continuous. Let $u_m \to u$ in X as $m \to \infty$. We have,

$$
\begin{aligned}
|\mathbf{B_3}(u_m,\varphi) - \mathbf{B_3}(u,\varphi)| &\leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u_m \nabla u_m) - \mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\, \omega_3\, dx \\
&= \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\, \omega_3\, dx \\
&\leq \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{r'}(\Omega,\omega_3)} \right) \|\, |\nabla\varphi|\, \|_{L^r(\Omega,\omega_3)} \\
&\leq \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{r'}(\Omega,\omega_3)} \right) \|\varphi\|_X,
\end{aligned}
$$

and, by Remark 2.4(b),

$$
\begin{aligned}
|\mathbf{B_4}(u_m,\varphi) - \mathbf{B_4}(u,\varphi)| &\leq \sum_{j=1}^n \int_\Omega |\mathcal{B}_j(x,u_m,\nabla u_m) - \mathcal{B}_j(x,u,\nabla u)||D_j\varphi|\, \omega_4\, dx \\
&= \sum_{j=1}^n \int_\Omega |G_j u_m - G_j u||D_j\varphi|\, \omega_4\, dx \\
&\leq \left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{s'}(\Omega,\omega_4)} \right) \|\, |\nabla\varphi|\, \|_{L^s(\Omega,\omega_4)} \\
&\leq M_2 \left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{s'}(\Omega,\nu_2)} \right) \|\, |\nabla\varphi|\, \|_{L^r(\Omega,\omega_3)}
\end{aligned}
$$

$$\leq \quad M_2\left(\sum_{j=1}^{n}\|G_j u_m - G_j u\|_{L^{s'}(\Omega,\nu_2)}\right)\|\varphi\|_X,$$

and we also have

$$\begin{aligned}
|\mathbf{B}_1(u_m,\varphi)-\mathbf{B}_1(u,\varphi)| &\leq \int_\Omega \left||\Delta u_m|^{p-2}\Delta u_m - |\Delta u|^{p-2}\Delta u\right||\Delta\varphi|\,\omega_1\,dx \\
&= \int_\Omega |Fu_m - Fu||\Delta\varphi|\,\omega_1\,dx \\
&\leq \|Fu_m - Fu\|_{L^{p'}(\Omega,\omega_1)}\,\|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\
&\leq \|Fu_m - Fu\|_{L^{p'}(\Omega,\omega_1)}\,\|\varphi\|_X,
\end{aligned}$$

and by Remark 2.4(a)

$$\begin{aligned}
|\mathbf{B}_2(u_m,\varphi)-\mathbf{B}_2(u,\varphi)| &\leq \int_\Omega \left||\Delta u_m|^{q-2}\Delta u_m - |\Delta u|^{q-2}\Delta u\right||\Delta\varphi|\,\omega_2\,dx \\
&= \int_\Omega |Gu_m - Gu||\Delta\varphi|\,\omega_2\,dx \\
&\leq \|Gu_m - Gu\|_{L^{q'}(\Omega,\omega_2)}\,\|\Delta\varphi\|_{L^q(\Omega,\omega_2)} \\
&\leq M_1\,\|Gu_m - Gu\|_{L^{q'}(\Omega,\omega_1)}\,\|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\
&\leq M_1\,\|Gu_m - Gu\|_{L^{q'}(\Omega,\omega_2)}\,\|\varphi\|_X,
\end{aligned}$$

and by Remark 2.4(b))

$$\begin{aligned}
|\mathbf{B}_5(u_m\varphi)-\mathbf{B}_5(u,\varphi)| &\leq \int_\Omega |\mathcal{H}(x,u_m,\nabla u_m)-\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\omega_5\,dx \\
&= \int_\Omega |Hu_m - Hu|\,|\varphi|\,\omega_5\,dx \\
&\leq \|Hu_m - Hu\|_{L^{z'}(\Omega,\omega_5)}\|\varphi\|_{L^z(\Omega,\omega_5)} \\
&\leq M_3\|Hu_m - Hu\|_{L^{z'}(\Omega,\omega_5)}\|\varphi\|_{L^p(\Omega,\omega_3)} \\
&\leq M_3\|Hu_m - Hu\|_{L^{z'}(\Omega,\omega_5)}\,C_\Omega\|\nabla\varphi\|_{L^p(\Omega,\omega_3)} \\
&\leq M_3\,C_\Omega\,\|Hu_m - Hu\|_{L^{z'}(\Omega,\omega_5)}\|\varphi\|_X,
\end{aligned}$$

for all $\varphi \in X$. Hence,

$$\begin{aligned}
&|\mathbf{B}(u_m,\varphi)-\mathbf{B}(u,\varphi)| \\
&\leq |\mathbf{B_1}(u_m,\varphi)-\mathbf{B_1}(u,\varphi)| + |\mathbf{B_2}(u_m,\varphi)-\mathbf{B_2}(u,\varphi)| \\
&+ |\mathbf{B_3}(u_m,\varphi)-\mathbf{B_3}(u,\varphi)| + |\mathbf{B_4}(u_m,\varphi)-\mathbf{B_4}(u,\varphi)|
\end{aligned}$$

$$
\begin{aligned}
&+|\mathbf{B_5}(u_m,\varphi)-\mathbf{B_5}(u,\varphi)| \\
&\leq \Bigg[\sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{r\,\prime}(\Omega,\omega_3)}+M_2\sum_{j=1}^{n}\|G_ju_m-G_ju\|_{L^{s\,\prime}(\Omega,\omega_4)} \\
&+\|Fu_m-Fu\|_{L^{p\,\prime}(\Omega,\omega_1)}+M_1\|Gu_m-Gu\|_{L^{q\,\prime}(\Omega,\omega_2)} \\
&+M_3C_\Omega\|Hu_m-Hu\|_{L^{z\,\prime}(\Omega,\omega_5)}\Bigg]\|\varphi\|_X.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\|Au_m-Au\|_* &\leq \sum_{j=1}^{n}\Big(\|F_ju_m-F_ju\|_{L^{r\,\prime}(\Omega,\omega_3)}+M_2\|G_ju_m-G_ju\|_{L^{s\,\prime}(\Omega,\omega_4)}\Big) \\
&+ \|Fu_m-Fu\|_{L^{p\,\prime}(\Omega,\omega_1)}+M_1\|Gu_m-Gu\|_{L^{q\,\prime}(\Omega,\omega_2)} \\
&+ M_3\,C_\Omega\|Hu_m-Hu\|_{L^{z\,\prime}(\Omega,\omega_5)}.
\end{aligned}
$$

Therefore, using (3.5), (3.8), (3.13), (3.14) and (3.16) we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous and this implies that A is hemicontinuous.

Therefore, by Theorem 3.1, the operator equation $Au=T$ has a unique solution $u\in X$ and it is the unique solution for problem (P).

Step 9. Estimates for $\|u\|_X$, if $2<r<\infty$. In particular, by setting $\varphi=u$ in Definition 2.4, we have

$$
\mathbf{B}(u,u)=\mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B_3}(u,u)+\mathbf{B_4}(u,u)+\mathbf{B_5}(u,u)=T(u). \tag{3.18}
$$

Hence, using (H3), (H7) and (H11) we obtain

$$
\begin{aligned}
\mathbf{B}(u,u) &= \mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B_3}(u,u)+\mathbf{B_4}(u,u)+\mathbf{B_5}(u,u) \\
&= \int_\Omega\langle\mathcal{A}(x,u,\nabla u),\nabla u\rangle\,\omega_3\,dx+\int_\Omega\langle\mathbf{B}(x,u,\nabla u),\nabla u\rangle\,\omega_4\,dx \\
&+ \int_\Omega|\Delta u|^p\,\omega_1\,dx+\int_\Omega|\Delta u|^q\,\omega_2\,dx+\int_\Omega\mathcal{H}(x,u,\nabla u)\,u\,\omega_5\,dx \\
&\geq \lambda_1\int_\Omega|\nabla u|^r\,\omega_3\,dx+\int_\Omega|\Delta u|^p\,\omega_1\,dx,
\end{aligned} \tag{3.19}
$$

and, since $\omega_3\in A_r$,

$$
\begin{aligned}
T(u) &= \int_\Omega f_0\,u\,dx+\sum_{j=1}^{n}\int_\Omega f_j\,D_ju\,dx \\
&\leq \|f_0/\omega_3\|_{L^{r\,\prime}(\Omega,\omega_3)}\|u\|_{L^r(\Omega,\omega_3)}+\Bigg(\sum_{j=1}^{n}\|f_j/\omega_3\|_{L^{r\,\prime}(\Omega,\omega_3)}\Bigg)\|\,|\nabla u|\,\|_{L^r(\Omega,\omega_3)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \left(C_\Omega \|f_0/\omega_3\|_{L^{r'}(\Omega,\omega_3)} + \sum_{j=1}^{n} \|f_j/\omega_3\|_{L^{r'}(\Omega,\omega_3)}\right)\|u\|_X \\
&= M\,\|u\|_X, \qquad (3.20)
\end{aligned}
$$

where $M = C_\Omega \|f_0/\omega_3\|_{L^{r'}(\Omega,\omega_3)} + \sum_{j=1}^{n} \|f_j/\omega_3\|_{L^{r'}(\Omega,\omega_3)}$. Hence in (3.18), using (3.19) and (3.20), we obtain

$$\lambda_1 \int_\Omega |\nabla u|^r\,\omega_3\,dx + \int_\Omega |\Delta u|^p\,\omega_1\,dx \le M\,\|u\|_X.$$

Therefore,

$$\|\Delta u\|^p_{L^p(\Omega,\omega_1)} \le M\,\|u\|_X \ \text{ and } \ \|\,|\nabla u|\,\|^r_{L^r(\Omega,\omega_3)} \le \frac{M}{\lambda_1}\,\|u\|_X.$$

By Young's inequality, we obtain

$$
\begin{aligned}
\|u\|_X &= \|\Delta u\|_{L^p(\Omega,\omega_1)} + \|\,|\nabla u|\,\|_{L^r(\Omega,\omega_3)} \\
&\le M^{1/p}\,\|u\|_X^{1/p} + \left(\frac{M}{\lambda_1}\right)^{1/r}\|u\|_X^{1/r} \\
&\le \frac{1}{p'}M^{p'/p} + \frac{1}{p}\,\|u\|_X + \frac{1}{r'}\left(\frac{M}{\lambda_1}\right)^{r'/r} + \frac{1}{r}\|u\|_X.
\end{aligned}
$$

Since $2 < r, p < \infty$, then $1/r + 1/p < 1$. Therefore, we obtain

$$\|u\|_X \le \gamma_{p,r}\left(\frac{1}{p'}M^{p'-1} + \frac{1}{r'}\big(M/\lambda_1\big)^{r'-1}\right),$$

where $\gamma_{p,r} = p\,r/(p\,r - p - r)$.

Example. Let $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 < 1\}$, the weight functions $\omega_1(x,y) = (x^2+y^2)^{5/2}$, $\omega_2(x,y) = (x^2+y^2)^{3/2}$, $\omega_3(x,y) = (x^2+y^2)^{-1/2}$, $\omega_4(x,y) = (x^2+y^2)^{-1/3}$ and $\omega_5(x,y) = (x^2+y^2)^{-1/3}$ ($\omega_3\in A_4$, $\omega_4\in A_3$, $\omega_1\in A_5$, $\omega_2\in A_3$, $p=5$, $q=3$, $r=4$, $s=z=3$), and the function

$$
\begin{aligned}
&\mathcal{A} : \Omega\times\mathbb{R}^2 \to \mathbb{R}^2 \\
&\mathcal{A}((x,y),\xi) = h_1(x,y)\,|\xi|^2\,\xi,
\end{aligned}
$$

where $h_1(x,y) = 2\,e^{(x^2+y^2)}$, and

$$
\begin{aligned}
&\mathcal{B} : \Omega\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2 \\
&\mathcal{B}((x,y),\eta,\xi) = g_2(x,y)\,|\xi|\,\xi,
\end{aligned}
$$

where $g_2(x,y) = 2 + \cos(x^2 + y^2)$ and

$$\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$$
$$\mathcal{H}((x,y),\eta,\xi) = h_3(x,y)\,\eta,$$

where $h_3(x,y) = 1 + \sin^2(xy)$. Let us consider the partial differential operator

$$\begin{aligned} Lu(x,y) &= \Delta\big[|\Delta u|^3 \Delta u\,\omega_1 + |\Delta u|\Delta u\,\omega_2\big] \\ &- \operatorname{div}\big(\mathcal{A}((x,y),u\nabla u)\,\omega_3(x,y) + \mathcal{B}((x,y),u,\nabla u)\omega_4(x,y)\big) \\ &+ \mathcal{H}((x,y),u,\nabla u)\,\omega_5(x,y). \end{aligned}$$

Therefore, by Theorem 1.1, the problem

$$(P)\begin{cases} Lu(x) = \dfrac{\cos(xy)}{\sqrt{x^2+y^2}} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right) \text{ in } \Omega, \\ u(x) = \Delta u(x) = 0 \text{ on } \partial\Omega \end{cases}$$

has a unique solution $u \in X = W_0^{1,4}(\Omega,\omega_3) \cap W_0^{2,5}(\Omega,\omega_1)$.

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