Qualitative Analysis of Quaternion Fuzzy Fractional Differential Equations with Ξ-Hilfer Fractional Derivative

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Abstract

In this work, we discuss the existence and stability of solution for a quaternion fuzzy fractional differential equation in the generalized regular fuzzy function space with Ξ-Hilfer fractional derivative. First of all, we give some definitions, theorems, and lemmas that are necessary for the understanding of the manuscript. Second of all, we give our first existence result, based on fixed point theorem and to deal with the uniqueness result. Next, this article is devoted to the investigation of the stability results. We employed a version of Piccard operator theorem to study the stability in the sense of Ulam-Hyers. In the end, we provide a couple of examples to illustrate our results.

Keywords: Quaternion fuzzy differential equation; Associate space; Fuzzy Ξ-Hilfer fractional derivative; Generalized regular function; Hyers-Ulam stability.

1 Introduction

Fractional calculus has been appearing in a wide range of fields, such as chemistry, economics, polymer rheology, and aerodynamics. This is due to the existence of many nice tools (see for instance [3, 4, 5, 6, 7]) that are not available in the classical calculus. Ξ-Hilfer fractional derivatives have been a considerable interest in the fractional calculus. The concept of fractional derivative of a function with respect to the another function Ξ suggest a new idea of fractional derivative. Many interesting results concerning the existence and stability of solutions by using various kinds of fixed-point techniques are available in the literature survey, one can refer to [8, 9, 10, 11, 12, 13] and references therein.

In this study, we investigate the existence and stability criteria for the solutions of the following initial value problem

$$\begin{cases} D^{\alpha,\beta,\Xi}_{0^+t}\mu = \sum_{j=1}^{3} A^{(j)}\frac{\partial \mathscr{A}}{\partial y_j} + B\mathscr{A} + C := \mathcal{L}(\mathscr{A}) \\ I^{(1-\alpha)(1-\beta),\Xi}_{0^+t}\mathscr{A}(0,y) = \varphi(y), \end{cases} \tag{1}$$

where Ω is a bounded in $\mathbb{R}^3$ and $y = (y_1, y_2, y_3) \in Omega.$ $D^{\alpha,\beta,\Xi}_{0^+t}$ is the Ξ-Hilfer fractional derivative of t; $t \in [0,T]$ is the time variable; $\mathscr{A} = \mathscr{A}(t,y)$ is a quaternion fuzzy-valued functions defined in $[0,T]\times\Omega$. $B = B(t,y)$, $A^{(j)} = A^{(j)}(t,y)$ and $C = C(t,y)$ are quaternion-valued function defined in $[0,T]\times\Omega$. The initial function $\varphi(y)$ is a generalized regular fuzzy function. We adopt some ideas from [14].

In 1989, Buckley [1] gave the first step towards the extension of fuzzy real numbers to complex fuzzy numbers. The quaternion membership function is given by $\mathscr{A} : \mathcal{H} \to [0,1]$ such that

$$\mathscr{A}(a + bi + cj + dk) = min\{\mathscr{A}_0(a), \mathscr{A}_1(b), \mathscr{A}_2(c), \mathscr{A}_3(d)\}$$

where $\mathscr{A}_i, i = 0, 1, 2, 3$ are real fuzzy numbers.

2 Preliminaries

Let us recall some basic definitions and notations of fractional calculus which is needed throughout this study.

Let $\mathsf{E}_K(\mathbb{R}^3)$ denote the family of all nonempty convex compact subsets of $\mathbb{R}^3$. The Hausdroff metric for $\mathscr{A}, \mathscr{B} \in \mathsf{E}_K(\mathbb{R}^3)$ is defined as

$$d(\mathscr{A}, \mathscr{B}) = inf\{\varepsilon | \mathscr{A} \subset N(\mathscr{B}, \varepsilon) \quad \text{and} \quad \mathscr{B} \subset N(\mathscr{A}, \varepsilon)\},$$

where $N(\mathscr{A}, \varepsilon) = \{a \in \mathbb{R}^3 : \|a - b\| < \epsilon \quad \text{for some} \quad a \in \mathscr{A}\}$. Throughout this paper, we denote $\Lambda := \{0, 1, 2, 3\}$ and $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$, where i, j, k are units of the real quaternion algebra $\mathcal{H}$.

Definition 2.1. *The quaternion membership function f is defined by*

$$f(V, \mathscr{A}) = e_0 f_0(V) + e_1 f_1(\mathscr{A}) + e_2 f_2(\mathscr{A}) + e_3 f_3(\mathscr{A}),$$

where V is to be interpreted as a set in fuzzy set of sets and $\mathscr{A} \in V$.
In particular, for $\mathscr{A} \in \mathbb{R}^3$, we have

$$f(\mathscr{A}) = e_0 f_0(\mathscr{A}) + e_1 f_1(\mathscr{A}) + e_2 f_2(\mathscr{A}) + e_3 f_3(\mathscr{A}),$$

where $f_0, f_1, f_2, f_3 : \mathbb{R}^3 \to [0, 1]$. Denote f by (f_0, f_1, f_2, f_3). Then, the $\overline{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$-level set of $f = (f_0, f_1, f_2, f_3)$ is defined as

$$[f]^{\overline{\alpha}} = [f_0]^{\alpha_0} \cap [f_1]^{\alpha_1} \cap [f_2]^{\alpha_2} \cap [f_3]^{\alpha_3}. \tag{2}$$

Denote J^n the set of all $\mathscr{B} : \mathbb{R}^n \to [0, 1]$ satisfying of the following conditions:

(i) $\mathscr{B}$ is normal, i.e., there exists $y_0 \in \mathbb{R}^n$ such that $\mathscr{B}(y_0) = 1$;

(ii) $\mathscr{B}$ is fuzzy convex, i.e., for all $a, b \in \mathbb{R}^n$, $\lambda \in [0, 1]$:

$$\mathscr{B}(\lambda a + (1 - \lambda) b) \geq min\{\mathscr{B}(a), \mathscr{B}(b)\};$$

(iii) $\mathscr{B}$ is upper semi-continuous;

(iv) $[\mathscr{B}]^0$ is compact.

Moreover, we define $\widehat{J}^{4n}$ as follows:

$\widehat{J}^{4n} = \{(\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3) \in J^n \times J^n \times J^n \times J^n | \exists t_0, s.t., \omega_l(t_0) = 1, l \in \Lambda\}$.
Then, for $\omega = (\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3) \in \widehat{J}^{4n}$, $[g]^{\overline{\alpha}} = \cap_{l \in \Lambda}[\mathscr{B}_l]^{\alpha_l} \in \mathsf{E}_K(\mathbb{R}^3)$ where $\alpha_l \in [0,1], l \in \Lambda$.
For $g, f \in \widehat{J}^{4n}$, where $f = (f_0, f_1, f_2, f_3)$ and $g = (g_0, g_1, g_2, g_3)$, and λ is a scalar, let

$$f + g = (f_0 + g_0, f_1 + g_1, f_2 + g_2, f_3 + g_3),$$
$$\lambda g = (\lambda g_0, \lambda g_1, \lambda g_2, \lambda g_3).$$

Let us define a metric $\mathcal{D} : J^n \times J^n \longrightarrow [0, \infty)$ by

$$\mathcal{D}(\mathscr{B}_1, \mathscr{B}_2) = sup\{d([\mathscr{B}_1]^r, [\mathscr{B}_2]^r) | r \in [0,1]\}, \tag{3}$$

where d is the Hausdorff distance. The metric space $(J^n, \mathcal{D})$ as a cone can be embedded isomorphically in a Banach space. However, $\mathcal{D}$ is not a suitable metric for our space of interst, $\widehat{J}^{4n}$, as we quickly see that linearity is violated. Instead, let us consider the product metric $\mathcal{D}^{'}$ on $J^{4n} = J^n \times J^n \times J^n \times J^n$. For $f = (f_0, f_1, f_2, f_3) \in J^{4n}$ and $g = (g_0, g_1, g_2, g_3) \in J^{4n}$, we modify the metric as $\mathcal{D}^{'} : J^{4n} \times J^{4n} \longrightarrow [0, \infty)$

$$\begin{aligned}\mathcal{D}^{'}(f, g) &= \mathcal{D}^{'}((f_0, f_1, f_2, f_3), (g_0, g_1, g_2, g_3)) \\ &= max_{l \in \Lambda}\{\mathcal{D}(f_l, g_l)\}. \end{aligned} \tag{4}$$

Then, the zero element on $\widehat{J}^{4n}$ is denoted as $\widehat{0}_4(y) = (\widehat{0}(y), \widehat{0}(y), \widehat{0}(y), \widehat{0}(y)) \in J^{4n}$. It is clear that $\mathcal{D}^{'}$ is a linearity preserving metric for J^{4n}. Since $\widehat{J}^{4n} \subset J^{4n}$, $\mathcal{D}^{'}$ is also metric for $\widehat{J}^{4n}$. Hence, $(\widehat{J}^{4n}, \mathcal{D})$ is a complete metric space. The metric space $(\widehat{J}^{4n}, \mathcal{D})$ can be embedded into a Banach space by the Arens-Eells theorem.
We introduce the strongly generalized differentiability in terms of the generalized Hukuhara difference. For $u, v \in \widehat{J}^{4n}$, if there exists $w \in \widehat{J}^{4n}$ such that $u = w + v$ or $v = u + (-1)w$, then we call w the difference of u and v and denote it as $u \ominus v = w$.
A fuzzy-valued function F defined on the bounded, simply connected domain $\Omega \subset \mathbb{R}^3$ is a mapping $F : \Omega \longrightarrow \widehat{J}^{4n}$, and F can be represented in a form $F = \sum_{j=0}^{3} e_j F_j(y)$. Its conjugate $\bar{F}$ is defined by

$$\bar{F} = e_0 F_0(y) \ominus \sum_{j=1}^{3} e_j F_j(y),$$

where $y = (y_1, y_2, y_3) \in \Omega$ and $F_j(y)$, $j = \Lambda$ are continuous fuzzy-valued functions.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain. The mapping $F : \Omega \to \widehat{\mathcal{J}}^{4n}$ is called strongly generalized partial derivative at $y = (y_1, y_2, y_3) \in \Omega$ if there exists some $\frac{\partial F}{\partial y_i} \in \widehat{\mathcal{J}}^{4n}$ such that*

(i) there exists the differences $F(\cdot, y_i + h, \cdot) \ominus F(\cdot, y_i, \cdot), F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i - h, \cdot)$ and

$$\begin{aligned} \frac{\partial F}{\partial y_i} &= \lim_{h\to 0^+} \frac{F(\cdot, y_i + h, \cdot) \ominus F(\cdot, y_i, \cdot)}{h} \\ &= \lim_{h\to 0^+} \frac{F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i - h, \cdot)}{h}, \end{aligned} \tag{5}$$

or

(ii) there exists the differences $F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i + h, \cdot), F(\cdot, y_i - h, \cdot) \ominus F(\cdot, y_i, \cdot)$ and

$$\begin{aligned} \frac{\partial F}{\partial y_i} &= \lim_{h\to 0^+} \frac{F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i + h, \cdot)}{-h} \\ &= \lim_{h\to 0^+} \frac{F(\cdot, y_i - h, \cdot) \ominus F(\cdot, y_i, \cdot)}{-h}, \end{aligned} \tag{6}$$

or

(iii) there exists the differences $F(\cdot, y_i + h, \cdot) \ominus F(\cdot, y_i, \cdot), F(\cdot, y_i - h, \cdot) \ominus F(\cdot, y_i, \cdot)$ and

$$\begin{aligned} \frac{\partial F}{\partial y_i} &= \lim_{h\to 0^+} \frac{F(\cdot, y_i + h, \cdot) \ominus F(\cdot, y_i, \cdot)}{h} \\ &= \lim_{h\to 0^+} \frac{F(\cdot, y_i - h, \cdot) \ominus F(\cdot, y_i, \cdot)}{-h}, \end{aligned} \tag{7}$$

or

(iv) there exists the differences $F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i + h, \cdot), F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i - h, \cdot)$ and

$$\begin{aligned} \frac{\partial F}{\partial y_i} &= \lim_{h\to 0^+} \frac{F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i + h, \cdot)}{-h} \\ &= \lim_{h\to 0^+} \frac{F(\cdot, y_i, \cdot) \ominus F(\cdot, y_i - h, \cdot)}{h}. \end{aligned} \tag{8}$$

In general, we have the following results on the connection between the strongly generalized partial differentiability of F and its endpoint functions F_l^α and F_r^α.

Let $F : \Omega \to \widehat{\mathcal{J}}^{4n}$ be a quaternion fuzzy function. If F is strongly generalized partial derivable at $y \in \Omega$, then we have the following case:

If F has strongly generalized partial derivative at $y \in \Omega$ in (i), then, for each $\alpha_i \in [0,1]$, F_{il} and F_{ir} are strongly generalized partial derivable functions at y and

$$\left[\frac{\partial F}{\partial y_i}\right]^{\alpha} = \left[\left(\frac{\partial F}{\partial y_i}\right)_l^{\alpha}, \left(\frac{\partial F}{\partial y_i}\right)_r^{\alpha}\right],$$

where

$$\left(\frac{\partial F}{\partial y_i}\right)_l^{\alpha} = \left[\left(\frac{\partial F}{\partial y_i}\right)_{0l}^{\alpha_0}, \left(\frac{\partial F}{\partial y_i}\right)_{1l}^{\alpha_1}, \left(\frac{\partial F}{\partial y_i}\right)_{2l}^{\alpha_2}, \left(\frac{\partial F}{\partial y_i}\right)_{3l}^{\alpha_3}\right] \tag{9}$$

and

$$\left(\frac{\partial F}{\partial y_i}\right)_r^{\alpha} = \left[\left(\frac{\partial F}{\partial y_i}\right)_{0r}^{\alpha_0}, \left(\frac{\partial F}{\partial y_i}\right)_{1r}^{\alpha_1}, \left(\frac{\partial F}{\partial y_i}\right)_{2r}^{\alpha_2}, \left(\frac{\partial F}{\partial y_i}\right)_{3r}^{\alpha_3}\right]. \tag{10}$$

Definition 2.3. *Let $F : \Omega \longrightarrow \widehat{\mathcal{J}}^{4n}$ be a continuous mapping. The fuzzy Ξ-type Riemann-Liouville integral of F is defined by*

$$({}^{RL}I_{0^+}^{\beta,\Xi}F)(y) = \frac{1}{\Gamma(\beta)}\int_0^{y_i} \Xi'(\tau)(\Xi(y_i) - \Xi(\tau))^{\beta-1}F(\cdot,\tau,\cdot)d\tau, \tag{11}$$

where $y \in \Omega$, $y_i > 0$, $0 < \beta < 1$.
Moreover, the Ξ-type Riemann-Liouville integral of a quaternion fuzzy-valued function F can be expressed as follows:

$$\left({}^{RL}I_{0^+}^{\beta,\Xi}F^{\alpha}\right)(y) = \left[\left({}^{RL}I_{0^+}^{\beta,\Xi}F_l^{\alpha}\right)(y), \left({}^{RL}I_{0^+}^{\beta,\Xi}F_r^{\alpha}\right)(y)\right],$$

where

$$\left({}^{RL}I_{0^+}^{\beta,\Xi}F_l^{\alpha}\right)(y) = \frac{1}{\Gamma(\beta)}\int_0^{y_i} \Xi'(\tau)(\Xi(y_i) - \Xi(\tau))^{\beta-1}F_l^{\alpha}(\cdot,\tau,\cdot)d\tau$$

and

$$\left({}^{RL}I_{0^+}^{\beta,\Xi}F_r^{\alpha}\right)(y) = \frac{1}{\Gamma(\beta)}\int_0^{y_i} \Xi'(\tau)(\Xi(y_i) - \Xi(\tau))^{\beta-1}F_r^{\alpha}(\cdot,\tau,\cdot)d\tau.$$

Definition 2.4. *The fuzzy Ξ-type Riemann-Liouville fractional derivative of order $n-1 < \beta < n$ for fuzzy-valued function F is defined by*

$$\left({}^{RL}D_{0^+}^{\beta,\Xi}F\right)(y) = \frac{1}{\Gamma(n-\beta)}\left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right)^n \int_0^{y_i} \Xi'(\tau)(\Xi(y_i) - \Xi(\tau))^{n-\beta-1}F(\cdot,\tau,\cdot)d\tau. \tag{12}$$

Similarly, we have

$$\left({}^{RL}D_{0^+}^{\beta,\Xi}F^{\alpha}\right)(y) = \left[\left({}^{RL}D_{0^+}^{\beta,\Xi}F_l^{\alpha}\right)(y), \left({}^{RL}D_{0^+}^{\beta,\Xi}F_r^{\alpha}\right)(y)\right], \tag{13}$$

where

$$\left({}^{RL}D_{0^+}^{\beta,\Xi}F_l^{\alpha}\right)(y) = \frac{1}{\Gamma(n-\beta)}\left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right)^n \int_0^{y_i} \Xi'(\tau)(\Xi(y_i)-\Xi(\tau))^{n-\beta-1}F_l^{\alpha}(\cdot,\tau,\cdot)d\tau. \tag{14}$$

and

$$\left({}^{RL}D_{0^+}^{\beta,\Xi}F_r^{\alpha}\right)(y) = \frac{1}{\Gamma(n-\beta)}\left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right)^n \int_0^{y_i} \Xi'(\tau)(\Xi(y_i)-\Xi(\tau))^{n-\beta-1}F_r^{\alpha}(\cdot,\tau,\cdot)d\tau. \tag{15}$$

Definition 2.5. *The fuzzy Ξ-type Caputo derivative of F for $n-1<\beta<n$ and $y\in\Omega$ is denoted as $\left({}^{C}D_{0^+}^{\beta,\Xi}F\right)(y)$ and defined by*

$$\left({}^{C}D_{0^+}^{\beta,\Xi}F\right)(y) = \frac{1}{\Gamma(n-\beta)}\int_0^{y_i} \Xi'(\tau)(\Xi(y_i)-\Xi(\tau))^{n-\beta-1}\left(\frac{1}{\Xi'(\tau)}\frac{\partial}{\partial \tau}\right)^n F(\cdot,\tau,\cdot)d\tau. \tag{16}$$

Then,

$$\left({}^{C}D_{0^+}^{\beta,\Xi}F^{\alpha}\right)(y) = \left[\left({}^{C}D_{0^+}^{\beta,\Xi}F_l^{\alpha}\right)(y), \left({}^{C}D_{0^+}^{\beta,\Xi}F_r^{\alpha}\right)(y)\right],$$

where

$$\left({}^{C}D_{0^+}^{\beta,\Xi}F_l^{\alpha}\right)(y) = \frac{1}{\Gamma(n-\beta)}\int_0^{y_i} \Xi'(\tau)(\Xi(y_i)-\Xi(\tau))^{n-\beta-1}\left(\frac{1}{\Xi'(\tau)}\frac{\partial}{\partial \tau}\right)^n F_l^{\alpha}(\cdot,\tau,\cdot)d\tau \tag{17}$$

and

$$\left({}^{C}D_{0^+}^{\beta,\Xi}F_r^{\alpha}\right)(y) = \frac{1}{\Gamma(n-\beta)}\int_0^{y_i} \Xi'(\tau)(\Xi(y_i)-\Xi(\tau))^{n-\beta-1}\left(\frac{1}{\Xi'(\tau)}\frac{\partial}{\partial \tau}\right)^n F_r^{\alpha}(\cdot,\tau,\cdot)d\tau. \tag{18}$$

Definition 2.6. *The fuzzy Ξ-Hilfer fractional derivative of order $\alpha\in[0,1]$ and $\beta\in(0,1)$ is defined as*

$$D_{0^+y_i}^{\alpha,\beta,\Xi}F(y) = I_{0^+y_i}^{\alpha(1-\beta),\Xi}\left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right)I_{0^+y_i}^{(1-\alpha)(1-\beta),\Xi}F(y) \tag{19}$$

for a function $F:\Omega\longrightarrow\widehat{J}^{4n}$ such that the expression on the right side exists.

Then

$$\begin{aligned}\left(D_{0^+y_i}^{\alpha,\beta,\Xi}F\right)(y) =& \left[\left(D_{0^+y_i}^{\alpha,\beta,\Xi}F_l^{\alpha}\right)(y), \left(D_{0^+y_i}^{\alpha,\beta,\Xi}F_r^{\alpha}\right)(y)\right]\\ =& \left[\left(I_{0^+y_i}^{\alpha(1-\beta),\Xi}\left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right)I_{0^+y_i}^{(1-\alpha)(1-\beta),\Xi}F_l^{\alpha}\right)(y),\right.\\ &\left.\left(I_{0^+y_i}^{\alpha(1-\beta),\Xi}\left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right)I_{0^+y_i}^{(1-\alpha)(1-\beta),\Xi}F_r^{\alpha}\right)(y)\right].\end{aligned} \tag{20}$$

Remark 1.

(i) When $\alpha = 0$ and $0 < \beta < 1$, the Ξ-Hilfer fractional derivative corresponds to the fuzzy Ξ-type Riemann-Liouville fractional derivative:

$$\begin{aligned} D_{0^+y_i}^{\alpha,\beta,\Xi}F(y) =& \left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right) I_{0^+y_i}^{(1-\beta),\Xi}F(y) \\ =& ^{RL}D_{0^+y_i}^{\alpha,\Xi}F(y). \end{aligned}$$

(ii) When $\alpha = 1$ and $0 < \beta < 1$, the fuzzy Ξ-Hilfer fractional derivative corresponds to the fuzzy Ξ-type Caputo fractional derivative:

$$\begin{aligned} D_{0^+y_i}^{\alpha,\beta,\Xi}F(y) =& I_{0^+y_i}^{(1-\beta),\Xi}\left(\frac{1}{\Xi'(y_i)}\frac{\partial}{\partial y_i}\right)F(y) \\ =& ^{C}D_{0^+y_i}^{\alpha,\Xi}F(y). \end{aligned}$$

Definition 2.7. *The fuzzy Dirac operator of F is defined as*

$$D(F) = \sum_{k=1,j=0}^{3} e_k e_j \frac{\partial F_j}{\partial y_k}.$$

Let η be a real number. The disturbed fuzzy Dirac operator is defined as

$$D_\eta \mathscr{A} = D\mathscr{A} + \eta\mathscr{A}.$$

Definition 2.8. *A fuzzy function $F : \Omega \to \widehat{J}^{4n}$ is called a generalized regular fuzzy function if it satisfies $D_\eta F = \widehat{0}_4$.*

Definition 2.9. *Assume that $\mathcal{L}(t, y, \mathscr{A})$ is a first order differential opeartor depending on the first order derivative $\frac{\partial\mathscr{A}}{\partial y_j}$ and $t, y, \mathscr{A}$, and that $l(t, y, u)$ is a differential operator on the time variate t. If $\mathcal{L}$ transforms solutions of $l\mathscr{A} = \widehat{0}_4$ into solutions of the same equations for fixed t (i.e. $l\mathscr{A} = \widehat{0}_4$ implies $l[\mathcal{L}\mathscr{A}] = \widehat{0}_4$), then $\mathcal{L}$ is called "associated" to l.*

Let $T : X \to Y$ be an abstract operator. Consider the fixed point equation

$$\tau = T(\tau), \quad \tau \in X \tag{21}$$

and the inequality

$$\mathcal{D}'(\tau, T(\tau)) \leq \epsilon. \tag{22}$$

Definition 2.10. *Assume that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function (continuous at 0 and $\phi(0) = 0$). If for each $\epsilon > 0$ and for each solution x^* of (22) there exists a solution y^* of the fixed point equation (21) such that*

$$\mathcal{D}^{'}(x^*, y^*) \leq \phi(\epsilon),$$

then the equation (21) is generalized Hyers-Ulam stable. For each $\tau \in \mathbb{R}^+$, if there exists $k > 0$ such that $\phi(\tau) := k\tau$, then the equation (21) is Hyers-Ulam stable.

3 Main results

The fixed points of the following operator equation (23) are the solutions of (1),

$$\begin{aligned} T(\mathscr{A}) :=& \mathscr{A}(t, y) \\ =& \frac{\varphi(y)}{\Gamma(\alpha(1-\beta)+\beta)}\Xi(t)^{(\alpha-1)(1-\beta)} + \frac{1}{\Gamma(\beta)}\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))\mathcal{L}(\mathscr{A})d\tau. \end{aligned} \tag{23}$$

Theorem 3.1. *Assume that $A^{(j)}(t, y)(j = 1, 2, 3)$, $B(t, y)$, $C(t, y)$ are quaternion-valued functions for $t \in [0, T]$. The operator $\mathcal{L}$ is associated with D_η, if the following hypotheses are satisfied:*

(1) $A_0^{(1)} = A_3^{(2)} = -A_2^{(3)}$,
$A_1^{(1)} = A_2^{(2)} = -A_3^{(3)}$,
$A_2^{(1)} = -A_1^{(2)} = A_0^{(3)}$,
$A_3^{(1)} = -A_0^{(2)} = -A_1^{(3)}$;

(2) $(DA^{(1)} + \eta A^{(1)} - 2B_1e_0)e_1 = (DA^{(2)} + \eta A^{(2)} - 2B_2e_0)e_2$
$=(DA^{(3)} + \eta A^{(3)} - 2B_3e_0)e_3$;

(3) $\eta DA^{(1)}e_1 + 2\eta^2 \sum_{j=1}^{3} A_j^{(1)}e_je_1 + 2\eta^2 A_1^{(1)}e_0 + DB + 2\eta(B_2e_2 + B_3e_3) = 0$;

(4) $D_\eta C = DC + \eta C = 0$ *for each* $t \in [0, T]$.

Proof. By Definition 2.13, if $D_\eta\mathscr{A} = \widehat{0}_4$ implies $D_\eta(\mathcal{L}\mathscr{A}) = \widehat{0}_4$, we can easily obtain that the operator $\mathcal{L}$ is associated with the operator D_η. It is easy to verify it, so we omit the proof here. □

Example 3.2. *If $\eta = 1$, $A^{(j)} = f(t, y_1, y_2, y_3)e_j (j = 1, 2, 3)$, $B = f(t, y_1, y_2, y_3)e_0$ and $C(t, y) = 0$, where real-valued function $f(t, y_1, y_2, y_3) \in C^2(\Omega)$ for each $t \in [0, T]$. Then the operator L is associated with the operator D_η.*
Moreover, we can get the interior estimate of a generalized fuzzy regular function by the associated function space theory.

Theorem 3.3. *Suppose that $\Omega_{s'} \subset \Omega_{s''}$ and $\bar{\Omega}_{s''} \subset \Omega$. Assume that $\mathscr{A}$ is a generalized fuzzy regular function and $m\Omega$ is the finite measure of $\Omega \subset \mathbb{R}^n$. We obtain the interior estimate of generalized fuzzy regular function*

$$\begin{aligned}\mathcal{D}^{'}\left(\frac{\partial \mathscr{A}}{\partial y_i}, \widehat{0}_4\right) \leq & \frac{\eta^2\left(\frac{3m\Omega}{4\pi}\right)^{\frac{1}{3}}[3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s'}, \partial\Omega_{s''})}\mathcal{D}^{'}(\mathscr{A}, \widehat{0}_4)\\ =&\gamma D^{'}(\mathscr{A}, \widehat{0}_4)\end{aligned} \tag{24}$$

Proof. Assume that $\mathscr{A}$ is a quaternion-valued function. It follows that, we have

$$\begin{aligned}\left\|\frac{\partial \mathscr{A}}{\partial y_i}\right\|_{s'} \leq & \frac{\eta^2\left(\frac{3m\Omega}{4\pi}\right)^{\frac{1}{3}}[3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s'}, \partial\Omega_{s''})}\|\mathscr{A}\|_{s''}\\ =&\gamma\|\mathscr{A}\|_{s''}.\end{aligned} \tag{25}$$

Now, for a generalized fuzzy regular function $\mathscr{A}$, we consider its endpoint functions $\mathscr{A}_l^\alpha$ and $\mathscr{A}_r^\alpha$. It is clear that v_l^α and v_r^α are generalized regular functions. We have

$$\left\|\frac{\partial \mathscr{A}_l^\alpha}{\partial y_i}\right\|_{s'} \leq \frac{\eta^2\left(\frac{3m\Omega}{4\pi}\right)^{\frac{1}{3}}[3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s'}, \partial\Omega_{s''})}\|\mathscr{A}_l^\alpha\|_{s''} \tag{26}$$

and

$$\left\|\frac{\partial \mathscr{A}_r^\alpha}{\partial y_i}\right\|_{s'} \leq \frac{\eta^2\left(\frac{3m\Omega}{4\pi}\right)^{\frac{1}{3}}[3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s'}, \partial\Omega_{s''})}\|\mathscr{A}_r^\alpha\|_{s''} \tag{27}$$

Moreover, we can obtain

$$\begin{aligned}\mathcal{D}^{'}\left(\frac{\partial \mathscr{A}}{\partial y_i}, \widehat{0}_4\right) =& \sup_{\alpha\in[0,1]}\left\{d\left(\left[\frac{\partial \mathscr{A}}{\partial y_i}\right]^\alpha, \widehat{0}_4\right)\right\}\\ =& \sup_{\alpha\in[0,1]}\left\{d\left(\left[\left(\frac{\partial \mathscr{A}}{\partial y_i}\right)_l^\alpha, \left(\frac{\partial \mathscr{A}}{\partial y_i}\right)_r^\alpha\right], \widehat{0}_4\right)\right\}\\ \leq&\frac{\eta^2\left(\frac{3m\Omega}{4\pi}\right)^{\frac{1}{3}}[3 + \frac{1}{2}(\frac{3m\Omega}{4\pi})^{\frac{1}{3}}]}{dist(\Omega_{s'}, \partial\Omega_{s''})}\sup_{\alpha\in[0,1]}\{d([\mathscr{A}_l^\alpha, \mathscr{A}_r^\alpha], \widehat{0}_4)\}\\ =&\gamma D^{'}(\mathscr{A}, \widehat{0}_4),\end{aligned} \tag{28}$$

where γ is a fixed constant.

□

For our subsequent results, we need the following hypotheses.

(H1) For $\mathscr{A} \in \widehat{J}^{4n}$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in [0,1] \times [0,1] \times [0,1] \times [0,1]$,

$$len_i(\mathscr{A}^{\alpha}) \geq len_i\bigg(\frac{\mathscr{A}_0^{\alpha}}{\Gamma(\alpha(1-\beta)+\beta)} t^{(\alpha-1)(1-\beta)} + \frac{1}{\Gamma(\beta)} \int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1} \mathcal{L}(\mathscr{A}) d\tau\bigg), \tag{29}$$

where $i \in \Lambda$ and $len_i(\mathscr{A}^{\alpha}) = \mathscr{A}_{ir}^{\alpha} - \mathscr{A}_{il}^{\alpha}$;

(H2) For $\mathscr{A} \in \widehat{J}^{4n}$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in [0,1] \times [0,1] \times [0,1] \times [0,1]$, $len_i(\mathscr{A}^{\alpha}(t, \cdot))$ is monotonous in t for $i \in \Lambda$, and $len_i(\mathscr{A}^{\alpha}(\cdot, y_j))$ is monotonus in y_j for $i, j \in \Lambda$;

(H3) For $\mathscr{A} \in \widehat{J}^{4n}$, $h > 0$, $i \in \Lambda$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in [0,1] \times [0,1] \times [0,1] \times [0,1]$, $\mathscr{A}_{il}^{\alpha_i}(t+h, \cdot) - \mathscr{A}_{il}^{\alpha_i}(t, \cdot)$ is nondecreasing in α_i and $\mathscr{A}_{ir}^{\alpha_i}(t+h, \cdot) - \mathscr{A}_{ir}^{\alpha_i}(t, \cdot)$ is nonincreasing in α_i;

(H4) For $\mathscr{A} \in \widehat{J}^{4n}$, $h > 0$, $i, j \in \Lambda$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in [0,1] \times [0,1] \times [0,1] \times [0,1]$, $\mathscr{A}_{il}^{\alpha_i}(\cdot, y_j+h) - \mathscr{A}_{il}^{\alpha_i}(\cdot, y_j)$ is nondecreasing in α_i and $\mathscr{A}_{ir}^{\alpha_i}(\cdot, y_j+h) - \mathscr{A}_{ir}^{\alpha_i}(\cdot, y_j)$ is nonincreasing in α_i;

Theorem 3.4. *Assume that $\mathcal{L}$ satisfies the hypotheses of Theorem 3.1 and the hypotheses $(H1)-(H4)$. The solution of the initial value problem (1) $\mathscr{A}(t, y)$ is in the conical domain $M_{\sigma} = \{(t, y) : y \in \Omega, t \in [0, \sigma] \cdot dist(y, \partial\Omega)\}$($\sigma$ is small enough), and is also generalized fuzzy regular for each t. Moreover, the fixed point equation $\mathscr{A} = T(\mathscr{A})$ is Hyers-Ulam stable.*

Proof. To prove this, we recall that any solution of the differential equation (1) must satisfy the Volterra equation

$$\mathscr{A}(t, y) = \frac{\varphi(y)}{\Gamma(\alpha(1-\beta)+\beta)} \Xi(t)^{(\alpha-1)(1-\beta)} + \frac{1}{\Gamma(\beta)} \int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1} \mathcal{L}(\mathscr{A}) d\tau.$$

Set that

$$T(\mathscr{A}) = \mathscr{A}(t, y) = \frac{\varphi(y)}{\Gamma(\alpha(1-\beta)+\beta)} \Xi(t)^{(\alpha-1)(1-\beta)} + \frac{1}{\Gamma(\beta)} \int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1} \mathcal{L}(\mathscr{A}) d\tau. \tag{30}$$

Then, we show that the operator T has a fixed point. It is clear that T maps $C([0,T]\times\Omega,\widehat{J}^{4n})$ to itself. Moreover, we find that

$$\begin{aligned}
\mathcal{D}^{'}(T(\mathscr{A})\ominus T(\nu),\widehat{0}_4) =&\mathcal{D}^{'}\Bigg(\frac{1}{\Gamma(\beta)}\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}\mathcal{L}(\mathscr{A})d\tau,\\
&\frac{1}{\Gamma(\beta)}\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}\mathcal{L}(\mathscr{B})d\tau\Bigg)\\
=&\frac{1}{\Gamma(\beta)}\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}\Bigg(\sum_{j=1}^{3}A^{(j)}\frac{1}{\Xi^{'}(y_j)}\frac{\partial}{\partial y_j}\mathcal{D}^{'}(\mathscr{A}\ominus\mathscr{B},\widehat{0}_4)\\
&+B\mathcal{D}^{'}(\mathscr{A}\ominus\mathscr{B},\widehat{0}_4)\Bigg)d\tau\\
\leq&\frac{1}{\Gamma(\beta)}(M+3\gamma N)\mathcal{D}^{'}(\mathscr{A}\ominus\mathscr{B},\widehat{0}_4)\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}d\tau\\
=&\frac{1}{\Gamma(\beta+1)}(M+3\gamma N)\Xi(t)^{\beta}\mathcal{D}^{'}(\mathscr{A}\ominus\mathscr{B},\widehat{0}_4)\\
=&\eta\mathcal{D}'(\mathscr{A}\ominus\mathscr{B},\widehat{0}_4), \qquad (31)
\end{aligned}$$

where $M=\|B\|$, $N=max_{j=1,2,3}\{\|A^{(j)}\|\}$.
We can choose a number $\tau>0$ such that

$$\eta=\frac{1}{\Gamma(\beta+1)}(M+3\gamma N)\tau^{\alpha}<1.$$

Then in the domain $M_\sigma=\{(t,y): y\in\Omega, t\in[0,\sigma].dist(y,\partial\Omega)\leq\tau\}$, T is a contraction mapping. Thus, by the Banach's fixed point theorem, we obtain the desired uniqueness of the solution of the differential equation. It follows that the operator T is a c-weakly Picard operator with the positive constant $c=\frac{1}{1-\eta}$ and the fixed point equation $\mathscr{A}=T(\mathscr{A})$ is Hyers-Ulam stable.
Moreover, the solution $\mathscr{A}(t,y)$ belongs to the associated space for each t. The solution $\mathscr{A}(t,y)$ is also generalized regular. □

Example 3.5. *Suppose that η is any real number, $C(t,y)\in C^1(\Omega,\mathcal{H})$ is any generalized regular function, and $A^{(1)}(t,y)\in C^2(\Omega,\mathcal{H})$ is any quaternion valued function for each $t\in[0,T]$. Suppose, further, that $A^{(2)}(t,y)=-A^{(1)}(t,y)e_3$, $A^{(3)}(t,y)=A^{(1)}(t,y)e_2$, $B(t,y)=-\eta A^{(1)}(t,y)e_1$. It is easy to verify that $\mathcal{L}$ is associated with $\mathcal{D}_\eta$ under the above conditions. Then by Theorem* 3.4*, there exists a unique solution of the initial value problem* (1)*, and the solution $\mathscr{A}(t,y)$ is also generalized fuzzy regular for each t. Moreover, the fixed point equation $\mathscr{A}=T(\mathscr{A})$ is Hyers-Ulam stable.*

In fact, the fixed point theorem plays a key role in the proof of Theorem 3.4, and this is called the fixed point method. By using the fixed point method, we continue to consider the existence and stability of the solution for the abstract Cauchy problem

$$\begin{cases} D_{0^+t}^{\alpha,\beta,\Xi}\mathscr{A}(t) = F(t,\mathscr{A}(t)), 0<\alpha,\beta<1, t\in[0,T], \\ \mathscr{A}(0)=\mathscr{A}_0, \end{cases} \tag{32}$$

where $\mathscr{A}(y)$ is a quaternion fuzzy function.

We denote the integral operator T by

$$\begin{aligned} T(\mathscr{A}(t)) =& \frac{\mathscr{A}_0}{\Gamma(\alpha(1-\beta)+\beta)}\Xi(t)^{(\alpha-1)(1-\beta)} \\ &+\frac{1}{\Gamma(\beta)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}F(\tau,\mathscr{A}(\tau))d\tau. \end{aligned} \tag{33}$$

Recall that $\mathscr{A}$ is a solution to the Cauchy problem (32) if and only if $\mathscr{A}$ is a solution to the integral equation (33). Moreover, $\mathscr{A}$ satisfies the integral equation (33) if and only if $\mathscr{A}$ satisfies the fixed point equation $\mathscr{A} = T(\mathscr{A})$. In other words, $\mathscr{A}$ is a solution to the Cauchy problem (32) if and only if $\mathscr{A}$ is a fixed point of the operator T.

$$\begin{aligned} T(\mathscr{A}) := \mathscr{A}(t) =& \frac{\mathscr{A}_0}{\Gamma(\alpha(1-\beta)+\beta)}\Xi(t)^{(\alpha-1)(1-\beta)} \\ &+\frac{1}{\Gamma(\beta)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}F(\tau,\mathscr{A}(\tau))d\tau. \end{aligned} \tag{34}$$

For our subsequent results, we need the following hypotheses.

(H5) There exists a constant M for which $D'(F(t,\mathscr{A}),\widehat{0}_4) < M$ holds for all $t\in I$ and all $\mathscr{A}\in\widehat{J}^{4n}$.

(H6) For $\mathscr{A}\in\widehat{J}^{4n}$ and $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in[0,1]\times[0,1]\times[0,1]\times[0,1]$,

$$\begin{aligned} len_i(\mathscr{A}^\alpha) \ge& len_i\Bigg(\frac{\mathscr{A}_0^\alpha}{\Gamma(\alpha(1-\beta)+\beta)}\Xi(t)^{(\alpha-1)(1-\beta)} \\ &+\frac{1}{\Gamma(\beta)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}F(\tau,\mathscr{A}^\alpha(\tau))d\tau\Bigg), \end{aligned} \tag{35}$$

where $i\in\Lambda$ and $len_i(\mathscr{A}^\alpha) = \mathscr{A}_{ir}^\alpha - \mathscr{A}_{il}^\alpha$,

(H7) For $\mathscr{A}\in\widehat{J}^{4n}$ and $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in[0,1]\times[0,1]\times[0,1]\times[0,1]$, $len_i(\mathscr{A}^\alpha(t))$ is monotonous in t for $i\in\Lambda$;

(H8) For $\mathscr{A}\in\widehat{J}^{4n}$, $h>0$, $i\in\Lambda$ and $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in[0,1]\times[0,1]\times[0,1]\times[0,1]$, $\mathscr{A}_{il}^\alpha(t+h)-\mathscr{A}_{il}^\alpha(t)$ is nondecreasing in α_i and $\mathscr{A}_{ir}^{\alpha_i}(t+h)-\mathscr{A}_{ir}^{\alpha_i}(t)$ is nonincreasing in α_i.

Theorem 3.6. *Assume that hypotheses* $(H5)-(H8)$ *are satisfied. Let* $F : I \times \widehat{J}^{4n} \to \widehat{J}^{4n}$ *be Lipschitz continuous and bounded, with Lipschitz constant L. Then, there exists a unique solution to the Cauchy problem* (1) *on a neighbourhood of* $a \in I$. *Moreover, the fixed point equation* $\mathscr{A} = T(\mathscr{A})$ *is Hyers-Ulam stable.*

Proof. By definition and hypotheses $(H5)-(H7)$, it is easy to verify that the Ξ-Hilfer derivative of $\mathscr{A}$ is well-defined. Let $[0,\epsilon]\times[\mathscr{A}_0-\delta,\mathscr{A}_0+\delta]\subset I\times\widehat{J}^{4n}$ be a compact subset on which F is defined. Then T to be well-defined, $D^{'}(T\mathscr{A}-\mathscr{A}_0,\widehat{0}_4)<\delta$ for all $\mathscr{A}\in[\mathscr{A}_0-\delta,\mathscr{A}_0+\delta]$, and

$$\begin{aligned}
D^{'}(T\mathscr{A}\ominus\mathscr{A}_0,\widehat{0}_4) =& D^{'}\left(\frac{1}{\Gamma(\beta)}\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}F(\tau,\mathscr{A}(\tau))d\tau,\widehat{0}_4\right)\\
\leq&\frac{1}{\Gamma(\beta)}\int_0^t \mathcal{D}^{'}\left(\Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}F(\tau,\mathscr{A}(\tau)),\widehat{0}_4\right)d\tau\\
\leq&\frac{M}{\Gamma(\beta)}\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}d\tau\\
\leq&\frac{M}{\Gamma(\beta+1)}\epsilon^{\beta}.
\end{aligned} \tag{36}$$

Hence we require $\frac{M}{\Gamma(\beta+1)}\epsilon^{\beta}<\delta$, i.e., we must pick $\epsilon>0$ such that $\epsilon<[\frac{\delta\Gamma(\beta+1)}{M}]^{\frac{1}{\beta}}$.
For $k=0,1,2,\ldots$, if we define $X_0(t)=\mathscr{A}_0$,

$$\begin{aligned}
X_{k+1}(t) =&\frac{\mathscr{A}_0}{\Gamma(\alpha(1-\beta)+\beta)}\Xi(t)^{(\alpha-1)(1-\beta)}\\
&+\frac{1}{\Gamma(\beta)}\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}F(\tau,X_k(\tau))d\tau.
\end{aligned} \tag{37}$$

Then,

$$\begin{aligned}
D^{'}(T(X_m\ominus X_n),\widehat{0}_4) =&\frac{1}{\Gamma(\beta)}D^{'}\Bigg(\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}(F(\tau,X_m(\tau))\\
&\ominus F(\tau,X_n(\tau)))d\tau,\widehat{0}_4\Bigg)\\
\leq&\frac{L}{\Gamma(\beta)}\left|\int_0^t \Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\beta-1}D^{'}(X_m(\tau)\ominus X_n(\tau),\widehat{0}_4)d\tau\right|\\
\leq&\frac{L}{\Gamma(\beta+1)}\epsilon^{\beta}D^{'}(X_m\ominus X_n,\widehat{0}_4).
\end{aligned} \tag{38}$$

If $\epsilon<min\{[\frac{\delta\Gamma(\beta+1)}{M}]^{\frac{1}{\beta}},[\frac{\Gamma(\beta+1)}{L}]^{\frac{1}{\beta}}\}$, the mapping is a contraction. In such a case, T is a contraction, and by the Banach fixed point theorem, T has a unique fixed point. Thus, there exists a unique $X^*\in C([0,\epsilon]\times[\mathscr{A}_0-\delta,\mathscr{A}_0+\delta])$ such that $T(X^*)=X^*$. One may construct this function by

$X^*(t) = lim_{k\to\infty} X_k(t)$. This function is the unique solution to the Cauchy problem (1) on the interval $[0, \epsilon]$, where $\epsilon < min\{[\frac{\delta\Gamma(\beta+1)}{M}]^{\frac{1}{\beta}}, [\frac{\Gamma(\beta+1)}{L}]^{\frac{1}{\beta}}\}$. It follows that the operator T is a c-weakly Picard operator with the positive constant $c = \frac{1}{1-\frac{L}{\Gamma(\beta+1)}\epsilon^{\beta}}$ and the fixed point equation $\mathscr{A} = T(\mathscr{A})$ is Hyers-Ulam stable. □

Example 3.7.

$$\begin{cases} D^{\alpha,\beta,\Xi}_{0^+t}\mathscr{A}(t) = F(t,\mathscr{A}(t)) = \sqrt{|\mathscr{A}(t)|+9} \ominus (i+j+k)sin(|\mathscr{A}(t)|) \\ \mathscr{A}(0) = \mathscr{A}_0, \quad 0<\alpha,\beta<1. \end{cases} \tag{39}$$

Let $\mathscr{A}, \mathscr{B} \in \widehat{\mathcal{J}}^{4n}$, and note that

$$\begin{aligned} D'(F(\mathscr{A}) \ominus F(\mathscr{B}), \widehat{0}_4) =& D'(\sqrt{|\mathscr{A}|^2+9} \ominus \sqrt{|\mathscr{B}|^2+9} \ominus (i+j+k)(sin(|\mathscr{A}|) \ominus sin(|\mathscr{B}|)), \widehat{0}_4) \\ \leq& D'(\sqrt{|\mathscr{A}|^2+9} \ominus \sqrt{|\mathscr{B}|^2+9}, \widehat{0}_4) + 3D'(sin(|\mathscr{A}|) \ominus sin(|\mathscr{B}|), \widehat{0}_4) \\ \leq& \frac{|\mathscr{A}|+|\mathscr{B}|}{\sqrt{|\mathscr{A}|^2+9} \ominus \sqrt{|\mathscr{B}|^2+9}} D'(|\mathscr{A}| \ominus |\mathscr{B}|, \widehat{0}_4) + 3D'(|\mathscr{A}| \ominus |\mathscr{B}|, \widehat{0}_4) \\ \leq& 4D'(\mathscr{A} \ominus \mathscr{B}, \widehat{0}_4) \end{aligned} \tag{40}$$

so F is Lipschitz continuous with Lipschitz constant $L = 4$. Furthermore, since $\mathscr{A} \in \widehat{\mathcal{J}}^{4n}$ by assumption, $\mathscr{A}(t) = (\mathscr{A}_0(t), \mathscr{A}_1(t), \mathscr{A}_2(t), \mathscr{A}_3(t))$ and $|\mathscr{A}_i| \leq 1$, $(i = 0, 1, 2, 3)$, hence $|F(\mathscr{A})| \leq \sqrt{13} + 3$. So F is continuous and bounded. Assume that hypotheses $(H5) - (H8)$ are satisfied. Theorem 3.6 unique local solution to (39). Moreover, the fixed point equation of (39) is Hyers-Ulam stable.

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